A220A0900: Operational Research Made Easy: Critical Approaches and Cultural Perspectives

Course at LUT University, Lappeenranta August 11-15 2025

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Andrea Saltelli

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Latest papers

Models and the common good

A short analysis of norms and counter-norms in modelling, discussing how models may misuse their epistemic authority for reasons of occasion, opportunity and interest.

Outline:
Narratives and counter-narratives,
State of exception,
Seeking Cinderella,
Reformation

Ranking the Rankers

Ranking the Rankers? We investigate using global sensitivity analysis the properties of the science-wide author databases' standardised citation indicators produced since 2015 by a team of researchers led by John P. A. Ioannidis.

Digital twins

Classroom

1) Write the equations for this transportation problem knowing that freight cost for each shipment is $100 plus 50 cents per mile. How much should be shipped from each plant to each of the distribution centers to minimize the total shipping cost?

9.1-2. The Childfair Company has three plants producing child push chairs that are to be shipped to four distribution centers. Plants 1, 2, and 3 produce 12, 17, and 11 shipments per month, respectively. Each distribution center needs to receive 10 shipments per month. The distance from each plant to the respective distributing centers is given below:

		Distance				
		Distribution Center				
		1	2	3	4	
Plant	1	800 miles	1,300 miles	400 miles	700 miles	12
	2	1,100 miles	1,400 miles	600 miles	1,000 miles	17
	3	600 miles	1,200 miles	800 miles	900 miles	11
		10	10	10	10	

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Classroom

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$$x_1 + x_2 + x_3 + x_4 = 12$$
$$x_5 + x_6 + x_7 + x_8 = 17$$
$$x_9 + x_{10} + x_{11} + x_{12} = 11$$

Plant constraints

$$x_1 + x_5 + x_9 = 10$$
$$x_2 + x_6 + x_{10} = 10$$
$$x_3 + x_7 + x_{11} = 10$$
$$x_4 + x_8 + x_{12} = 10$$

Distribution centres constraints

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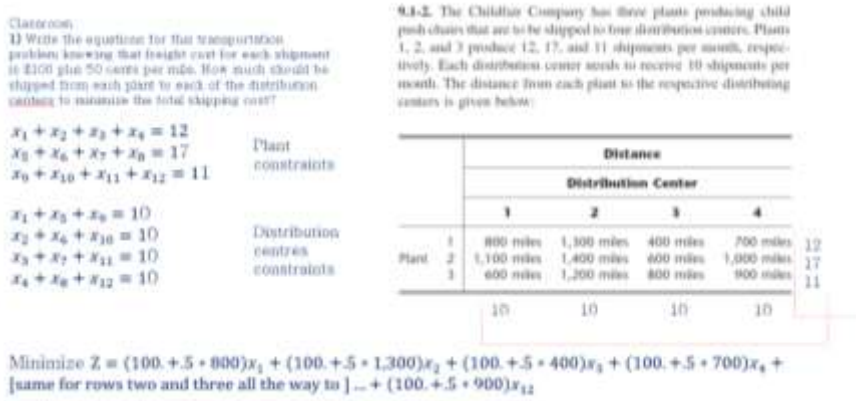
Minimize $Z = (100. + .5 * 800)x_1 + (100. + .5 * 1{,}300)x_2 + (100. + .5 * 400)x_3 + (100. + .5 * 700)x_4 +$ [same for rows two and three all the way to] $\ldots + (100. + .5 * 900)x_{12}$



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$x_1 + x_5 + x_9 = 10$
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[same for rows two and three all the way to] ... $+ (100. + .5 * 900)x_{12}$

Solution with Excel Solver

Z=20,200

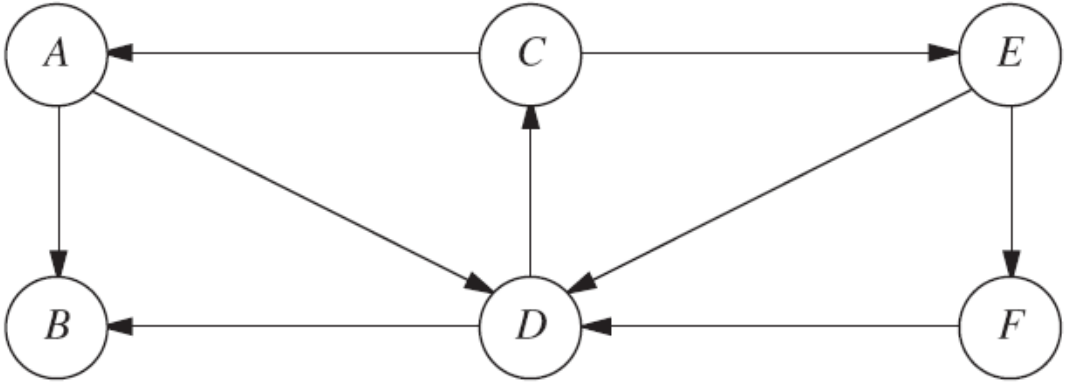
See it at

https://www.andreasaltelli.eu/file/repository/Childfair_Company_Revised.xlsx

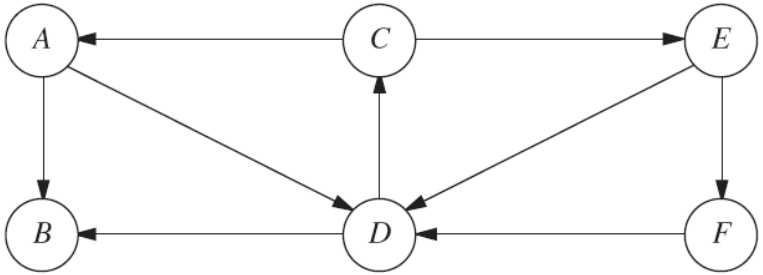
79									
80	*Decision var. &*		Dist. Centers						
81	*Constraints*		1	2	3	4	Shipments		Available
82	Plant	1	0	0	2	10	12	<=	12
83		2	0	9	8	0	17	<=	17
84		3	10	1	0	0	11	<=	11
85	Received		10	10	10	10			
86			>=	>=	>=	>=			
87	Required		10	10	10	10			
88									
89	**Objective**								
90	Minimize TCs		20200						

Homework

2) Consider the following directed network (Hillier 9.2-1)



(a) Find a directed path from node A to node F, and then identify three other undirected paths from node A to node F.
(b) Find three directed cycles. Then identify an undirected cycle that includes every node.
(c) Identify a set of arcs that forms a spanning tree.
(d) Use the process illustrated in Fig. 10.3 to grow a tree one arc at a time until a spanning tree has been formed. Then repeat this process to obtain another spanning tree. [Do not duplicate the spanning tree identified in part (c).]



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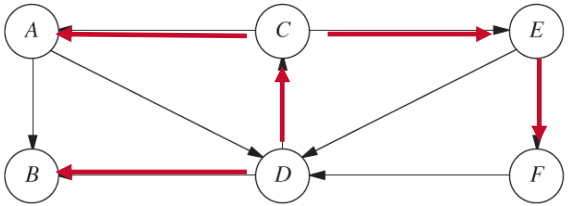
(a) $A \rightarrow D \rightarrow C \rightarrow E \rightarrow F$ directed

$A \rightarrow C \rightarrow D \rightarrow F$ undirected
$A \rightarrow B \rightarrow D \rightarrow F$ undirected
$A \rightarrow D \rightarrow F$ undirected

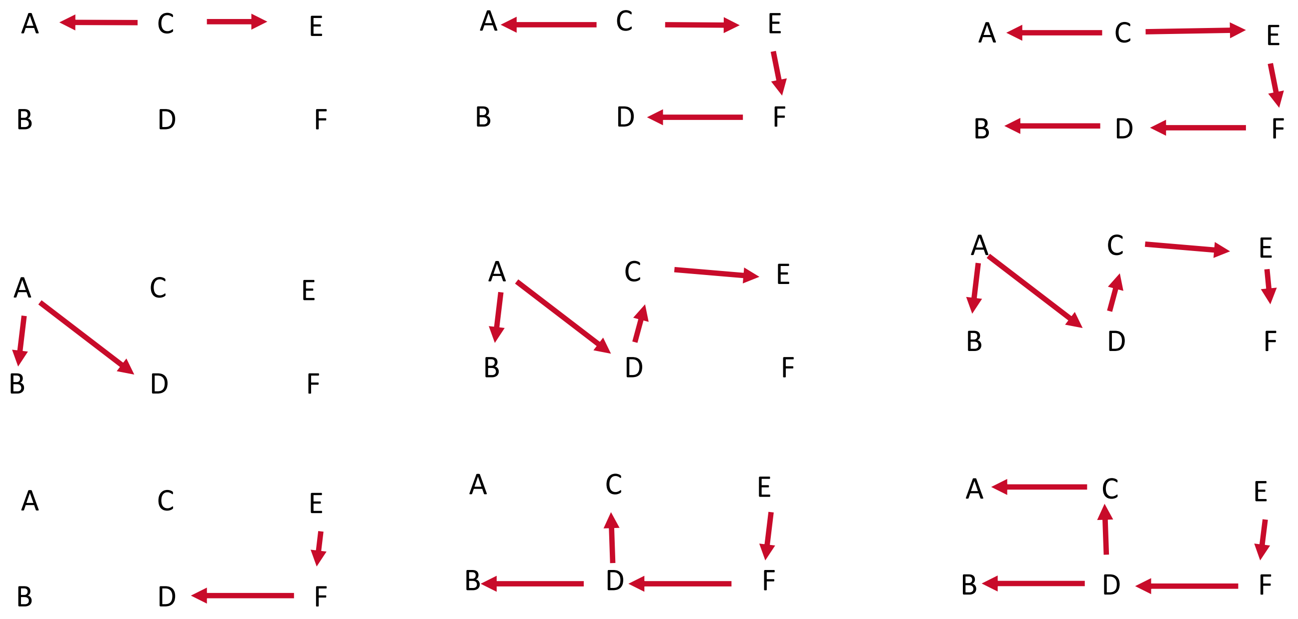
(b) $A \rightarrow D \rightarrow C \rightarrow A$ directed
$D \rightarrow C \rightarrow E \rightarrow D$ directed
$E \rightarrow F \rightarrow D \rightarrow C \rightarrow E$ directed

$A \rightarrow B \rightarrow D \rightarrow F \rightarrow E \rightarrow C \rightarrow A$ undirected all nodes

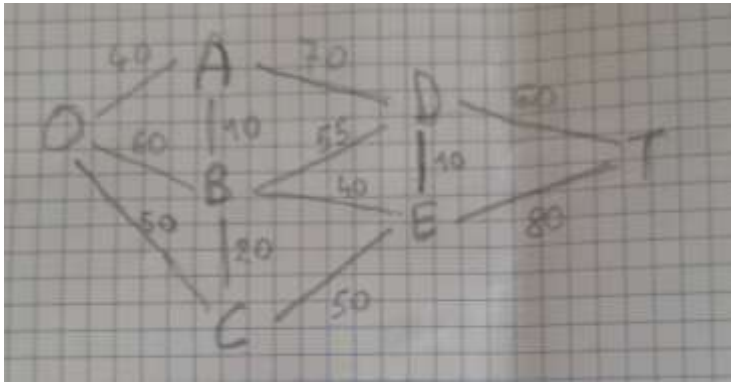
(c) Spanning tree



(c) Three different spanning trees



Homework 3) You need to take a trip by car to another town that you have never visited before. Therefore, you are studying a map to determine the shortest route to your destination. Depending on which route you choose, there are five other towns (call them A, B, C, D, E) that you might pass through on the way. The map shows the mileage along each road that directly connects two towns without any intervening towns. These numbers are summarized in the following table, where a dash indicates that there is no road directly connecting these two towns without going through any other towns. Formulate this problem as a shortest-path problem by drawing a network where nodes represent towns, links represent roads, and numbers indicate the length of each link in miles.



Town	Miles between Adjacent Towns					
	A	B	C	D	E	Destination
Origin	40	60	50	—	—	—
A		10	—	70	—	—
B			20	55	40	—
C				—	50	—
D					10	60
E						80

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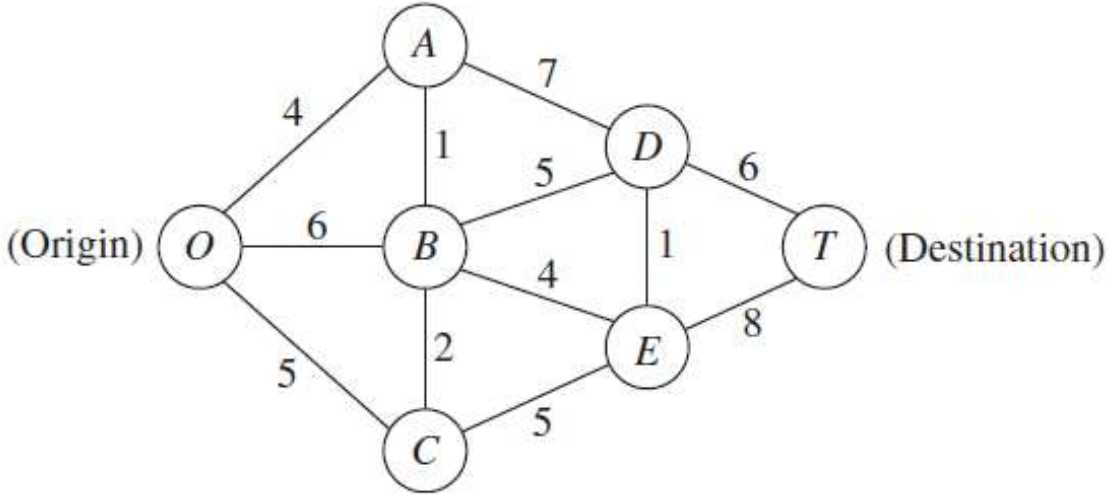
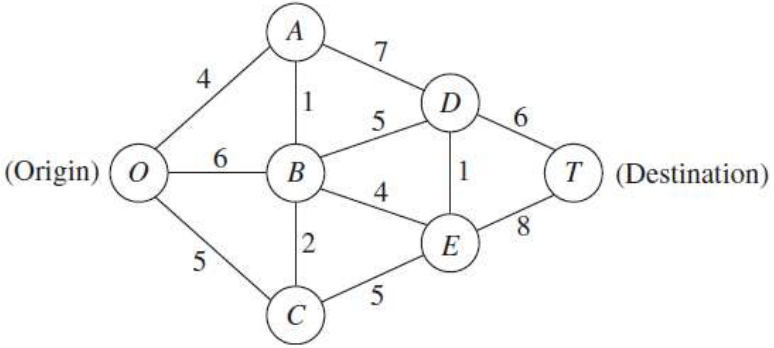
O-A-B-E-D-T?

Homework 4) Find shortest path from *O* to *T*, first visually then using the table method and backward recursion studied in Lesson 4 (Hillier 10.3-4); the first row of the table in give below.



n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	4	*A*	4	*OA*

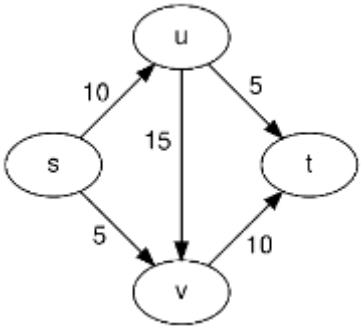
Homework 4)

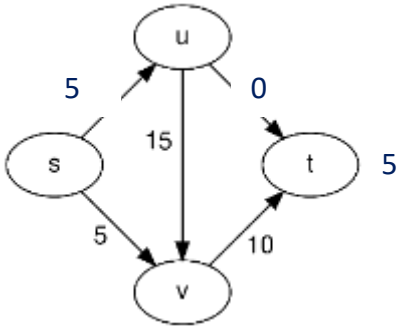


n	added	dest	totdist	min	mindist	last connection
1	O	A	4	A	4	OA
2	O	C	5	C	5	OC
	A	B	4+1	B		AB
3	A	D	11			
	B	E	4+4	E	9	BE
	C	E	10			
4	B	D	10	D	10	BD
	E	D	10			ED
5	DT		16	T	16	DT
	ET		17			

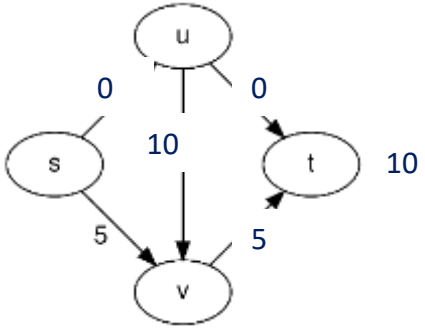
TDEBAO
TDBAO
↳ OABEDT
OABDT

5) Solve the maximum flow problem from node s to node t for this network showing your steps. Numbers on the links are capacities.

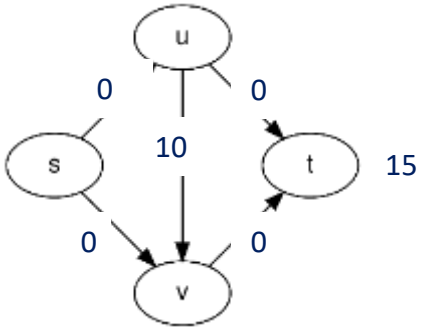




Move 5 via s-u-t



Move 5 via s-u-v-t



Move 5 via s-v-t

Solution=15 units

6) Launching two coins 6 times which is the probability of getting two heads exactly twice.

Solution: $\binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{6-2} = 15\,\frac{3^4}{4^6} = .297$

Experiment: launching 2 coins

Space of the events {HH,TT,HT,TH}

Event being sought HH; P(H)=1/4

Number of replications of the experiment 6

Occurrences of the event 2

$$\binom{6}{2} = \frac{6!}{2! * 4!} = \frac{6 * 5}{2} = 15$$

7) The Medequip Company produces precision medical diagnostic equipment at two factories. Three medical centers have placed orders for this month's production output. The table below shows what the cost would be for shipping each unit from each factory to each of these customers. Also shown are the number of units that will be produced at each factory and the number of units ordered by each customer.

From \ To	Unit Shipping Cost			
	Customer 1	Customer 2	Customer 3	Output
Factory 1	$600	$800	$700	400 units
Factory 2	$400	$900	$600	500 units
Order size	300 units	200 units	400 units	

A decision now needs to be made about the shipping plan for how many units to ship from each factory to each customer.

(a) Formulate a linear programming model for this problem (without solving it).

(b) Knowing that the solution is

From Factory 1, ship 200 units to Customer 2 and 200 units to Customer 3.

From Factory 2, ship 300 units to Customer 1 and 200 units to Customer 3.

Check that this solution satisfies the constraints you have written.

Solution: Let x_{ij} be the cost of shipping from factory i to customer j , then

Minimize

$600x_{11}+ 800x_{12}+ 700x_{13}+ 400x_{21}+ 900x_{22}+ 600x_{23}$

Subject to

$x_{11}+ x_{12}+ x_{13}=400$

$x_{21}+ x_{22}+ x_{23}=500$

$x_{11}+ x_{21}=300$

$x_{12}+ x_{22}=200$

$x_{13}+ x_{23}=400$

and $x_{ij}\geq 0$ for all i,j

Verify:

0+200+200=400

300+0+200=500

0+300=300

200+0=200

200+200=400

Given solutions

$x_{11}, x_{12}, x_{13}=0, 200, 200$

$x_{21}, x_{22}, x_{23}=300, 0, 200$

$Z=800*200+ 700*200+ 400*300+ 600*200=$
$10{,}000*(16+ 14+ 12+ 12)=540{,}000$

8) In a sample survey, 1,800 senior citizens were asked whether or not they have ever been victimized by a dishonest telemarketer. The following table gives the responses by age group.

Suppose one person is randomly selected from these senior citizens. Find the following probabilities.

a) P(have been victimized or B)

b) P(have never been victimized or C)

			Have Been Victimized	Have Never Been Victimized	
Age	60–69	(A)	106	698	804
	70–79	(B)	145	447	592
	80 or over	(C)	61	343	404
			312	1488	1800

Solution

$$P(V \cup B) = P(V) + P(B) - P(V \cap B) = \frac{312 + 592 - 145}{1800} = 0.42$$

$$P(NV \cup C) = P(NV) + P(C) - P(NV \cap C) = \frac{1488+404-343}{1800} = 0.86$$

In this set of slides:

15 Integer programming (continued)
16 Nonlinear programming

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15.

Integer Programming (continued)

Solution via branch and bound. Take home points.
Hillier 2014, chapter 12.

More tricks with binary variables. From Hillier 10th edition, section 12.3
INNOVATIVE USES OF BINARY VARIABLES IN MODEL FORMULATION

When one of two constraints must hold, for example

$$3x_1 + 5x_2 - 7x_3 \leq 12$$

or

$$4x_1 + 2x_2 + x_3 \leq 15$$

But not both we can use an auxiliary binary variable y and impose

$$3x_1 + 5x_2 - 7x_3 \leq 12 + My$$
$$4x_1 + 2x_2 + x_3 \leq 15 + M(1 - y)$$
$$x_i \geq 0$$
y binary

Where M is the usual large number.
If $y = 0$ the first constraint holds, if $y = 1$ the second

More tricks with binary variables.

Another case where IP comes handy is when a problem needs one out of a set of constraints to hold. Old example:

TABLE 3.1 Data for the Wyndor Glass Co. problem

Plant	Production Time per Batch, Hours: Product 1	Product 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3,000	$5,000	

OLD PROBLEM

Maximize $Z = 3x_1 + 5x_2$

Subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Imagine we change one constraint so that in plant three there are three possible available times, depending on other possible manufactures so $3x_1 + 2x_2$ must now be equal to either 6 or 12 or 18

Tip: use three binary variable y_1, y_2, y_3

$3x_1 + 2x_2 \leq 6y_1 + 12y_2 + 18y_3$

$y_1 + y_2 + y_3 = 1$ and …?

y_1, y_2, y_3 binary

This is now a mixed integer programming problem (MIP)

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"It is common for an IP algorithm to begin by applying the simplex method to the LP relaxation to check whether this fortuitous outcome has occurred"

This may or may not work see e.g. the simple example

Maximize $Z = x_2$ (this mean x_2 as high as possible) subject to

$-x_1 + x_2 \leq \frac{1}{2}$

$x_1 + x_2 \leq \frac{7}{2}$

and

$x_1 \geq 0, x_2 \geq 0$

x_1, x_2 integers

Find graphically the linear solution of this problem

I.e. removing this constraint

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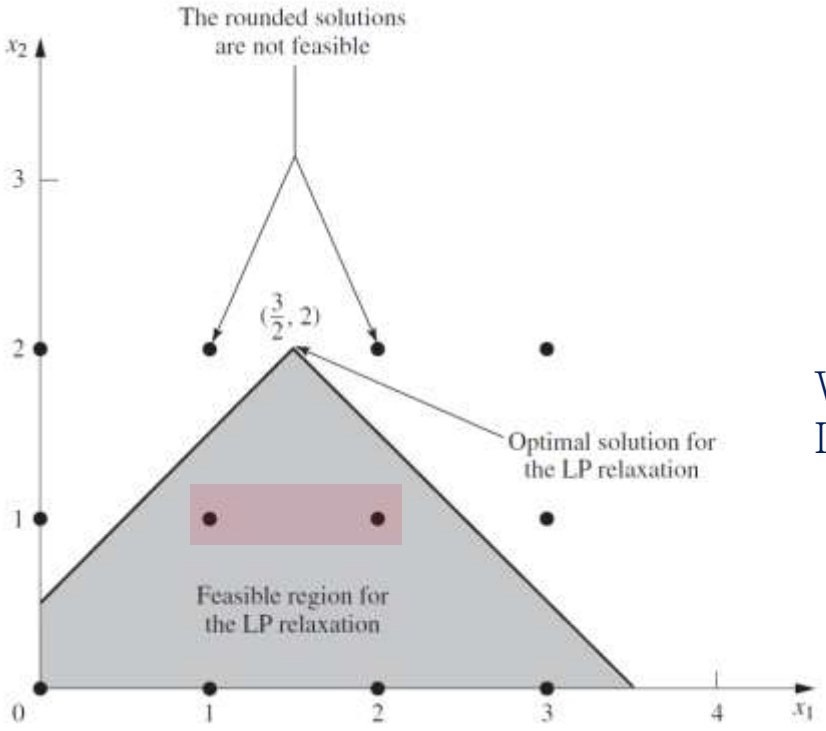


FIGURE 12.2
An example of an IP problem where the optimal solution for the LP relaxation cannot be rounded in any way that retains feasibility.

Which is instead the IP solutions?

Another case where the relaxation solution can be **not OK**

Maximize $Z = x_1 + 5x_2$ subject to

$x_1 + 10x_2 \leq 20$

$x_1 \leq 2$

and

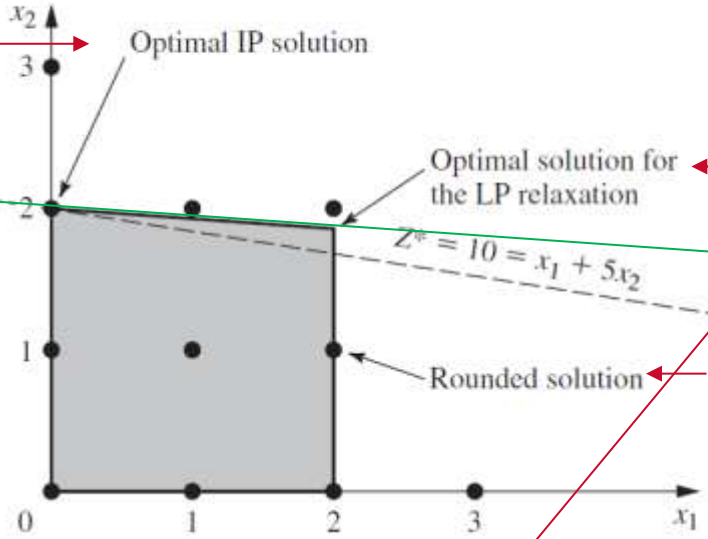
$x_1 \geq 0, x_2 \geq 0$

x_1, x_2 integers

Find graphically the linear solution of this problem …

… i.e. removing this constraint

$x_1 = 0,$
$x_2 = 2$
$Z = 10$



Line $x_1 + 10x_2 = 20$

$x_1 = 2,$
$x_2 = \frac{9}{5}$
$Z = 11$

FIGURE 12.3
An example where rounding the optimal solution for the LP relaxation is far from optimal for the IP problem.

$x_1 = 2,$
$x_2 = 1$
$Z = 7$

Maximize $Z = x_1 + 5x_2$ subject to

$x_1 + 10x_2 \leq 20$

$x_1 \leq 2$

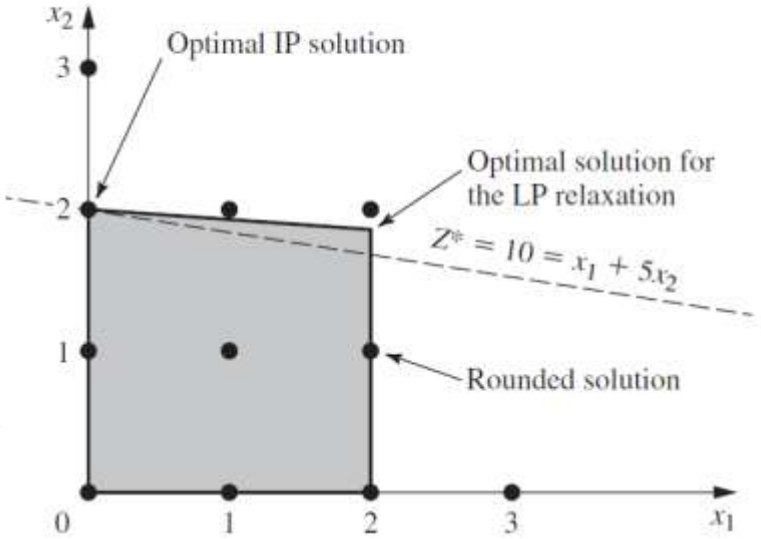
and

$x_1 \geq 0, x_2 \geq 0$

Did we violate the rule that the LP solution is an upper bound for the IP solution?

When there are may dimensions checking that the relaxation solution is OK can be tricky;

Here we have only **7 integer** points in the feasible region, but the number of points grows exponentially with the number of dimensions



In many dimensions better use metaheuristic method (such as genetic algorithms, more later) that also work for nonlinear problems.

But there are IP problems whose structure guarantees an integer solution; remember the Transportation Problem (Section 12);

The integer solutions property: For transportation problems where every supply s_i and demand d_i have an integer value, all basic feasible (BF) solutions (including an optimal one) also have integer values

TABLE 9.3 Constraint coefficients for P & T Co.

	Coefficient of:												
	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	
A =	1	1	1	1									Cannery constraints
					1	1	1	1					
									1	1	1	1	
	1				1				1				Warehouse constraints
		1				1				1			
			1				1				1		
				1				1				1	

But there are IP problems whose structure guarantees an integer solution; remember from the section on Transportation Problem (Section 12);

Other special cases are the assignment problem, the shortest-path problem, and the maximum flow problem

Source: Wikipedia Commons

Charles Chaplin's Modern Times, source http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

Ramon Casas and Pere Romeu on a Tandem, Barcelona. Source: Wikipedia Commons

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Level of difficulty of LP versus IP

	Difficulty of LP problem	Difficulty of IP problem
Source	Number of constraints	Number of integer variables
		Binary or general integer?
		Special form?

Source: https://www.dreamstime.com/illustration/accountant.html

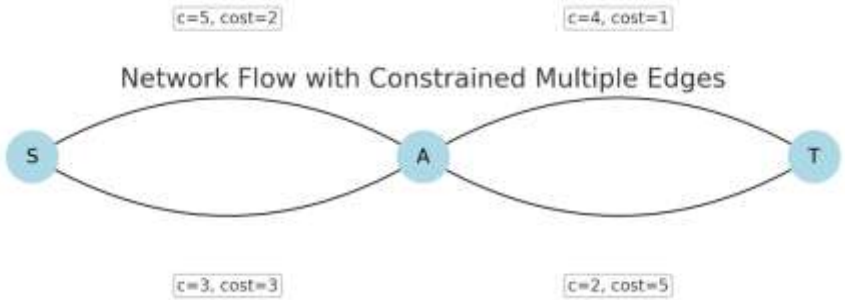
Such as Combinatorial Explosion: ➔ Knapsack or Traveling Salesman Problems (TSP) grow exponentially with problem size.

Level of difficulty of LP versus IP

	Difficulty of LP problem	Difficulty of IP problem
Source	Number of constraints	Number of integer variables
		Binary or general integer?
		Special form?

Such as Combinatorial Explosion: ➔ Knapsack or Traveling Salesman Problems (TSP) grow exponentially with problem size.

Coupling between Constraints: when the constraints are tightly coupled or interdependent, small changes in one variable can drastically affect the rest of the solution ➔ Network flow problems with constrained multiple edges.



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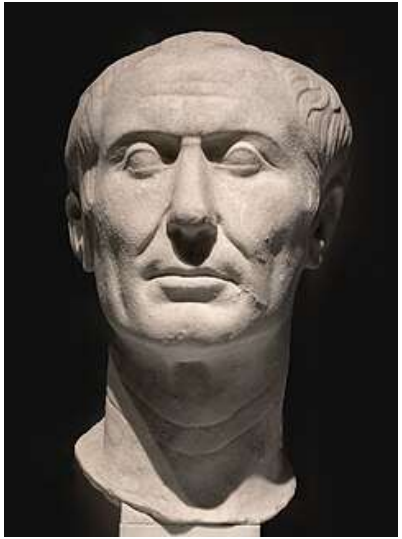
An illustrative algorithm for integer programming: the branch-and-bound technique:

"The basic concept underlying the branch-and-bound technique is to divide and conquer" (Hillier, p. 502)



Julius Caesar
(100-44 BC)

Source: Wikipedia commons

Caesar's '*Divide et impera*')…

… says Macchiavelli

Niccolò Machiavelli
(1469-1527)

Source: Wikipedia commons

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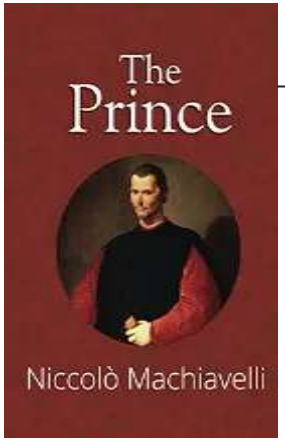
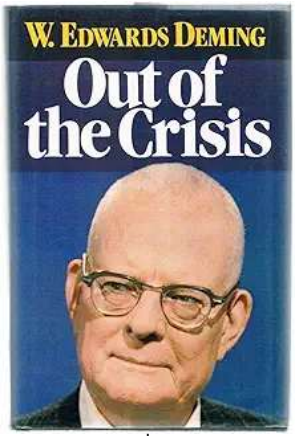
Digression: Macchiavelli?



Realpolitik & Power Dynamics, Ethics vs. Effectiveness ("better to be feared than to be loved": Instrumental rationality instead of normative ethics, Decision-Making under Uncertainty (*fortuna* (=chance/luck) versus *virtù* (=skill/agency)), Human Nature and Motivation



Understanding variation using statistical methods, theory of knowledge, psychology, cultural shift and active leadership to break down barriers between departments … ➔ total quality management, Six sigma …

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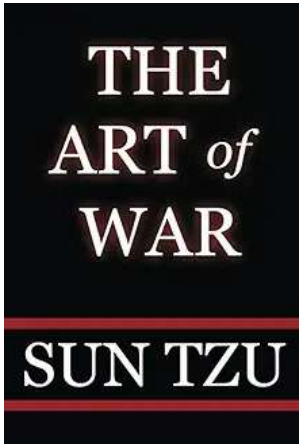
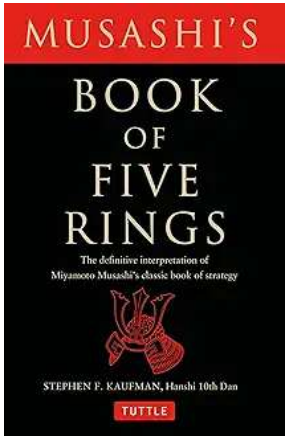
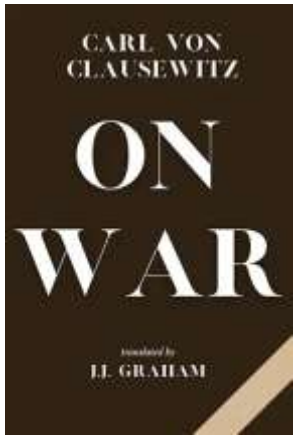
Digression: a manager may want to have read these as well…



Choice of time



Strategy as *Dō*, way



Higher purpose : “War is merely the continuation of policy by other means.”

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Back to out prototype example: building or not building?

TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	$9 million	$6 million
2	Build factory in San Francisco?	x_2	$5 million	$3 million
3	Build warehouse in Los Angeles?	x_3	$6 million	$5 million
4	Build warehouse in San Francisco?	x_4	$4 million	$2 million
			Capital available:	$10 million

The choice is if building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building **at most one** new warehouse, but the choice of location is restricted to a city where a new factory is being built.

TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	$9 million	$6 million
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3	Build warehouse in Los Angeles?	x_3	$6 million	$5 million
4	Build warehouse in San Francisco?	x_4	$4 million	$2 million
			Capital available:	$10 million

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$

Subject to:

$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$

$-x_1 + x_3 \leq 0$

$-x_2 + x_4 \leq 0$

$x_3 + x_4 \leq 1$

and

x_j binary for $j = 1,2,3,4$

If we apply LP relaxation replacing x_j binary for $j = 1,2,3,4$ with

$x_j \geq 0$ for $j = 1,2,3,4$

We obtain $x_1, x_2, x_3, x_4 = \left(\frac{5}{6}, 1, 0, 1\right)$ with $Z = 16.5$; is this a feasible solution?

We **round this to 16** and keep it as an upper bound for the IP problem. Why?

Because all coefficients in the objective function are integer, so all integer solutions must have an integer value for Z.

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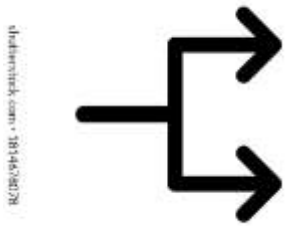
One method to solve IP problems: the branch-and-bound technique

- Branching (split the problem in two branches)
- Bounding (seek for a local optima for Z)
- Fathoming (Resolving the branching at fathomed the node)



Source: https://www.123rf.com/

Source: https://thesaurus.plus/synonyms/fathomed

- Branching (split the problem in two branches), rewriting

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

$x_1 = 0$

$x_1 = 1$

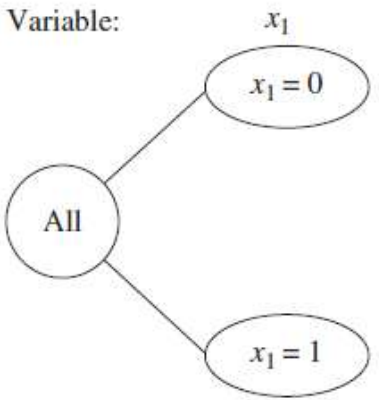
Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \leq 10$
$x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

(x_1 now disappears)

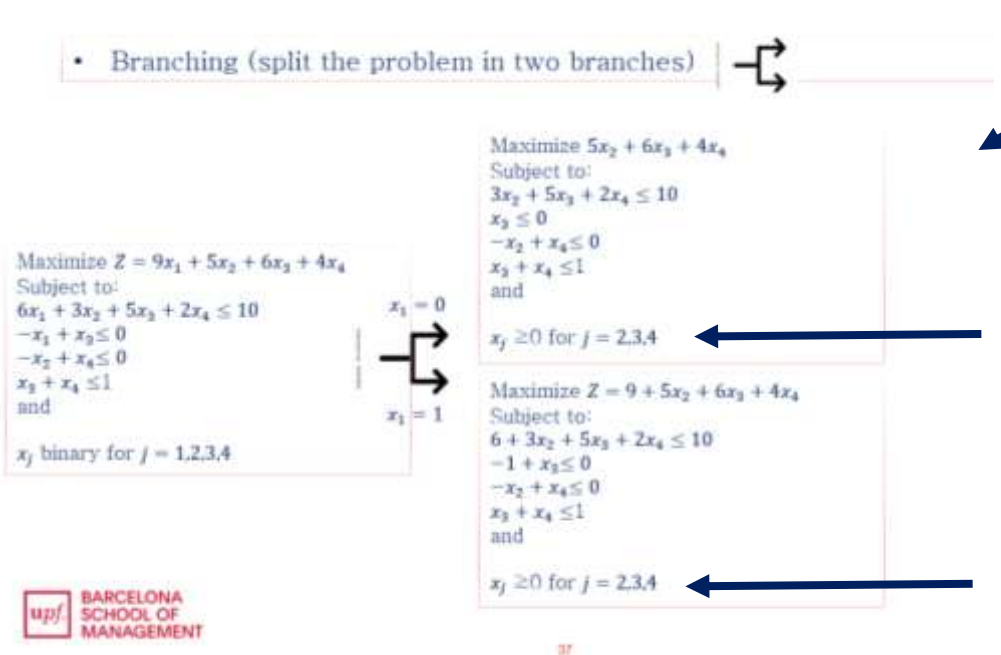
Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$



- Branching (split the problem in two branches)

We are splitting following the order of the variables, i.e. here starting by x_1.



The two subproblems are treated as linear instead of integer

- Bounding (seek for local optima for Z)

- Branching (split the problem in two branches)

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

$x_1 = 0$

$x_1 = 1$

Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \leq 10$
$x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

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Linear programming applied to these solutions yields

$x_1, x_2, x_3, x_4 = (0,1,0,1)$ with $Z = 9$

$x_1, x_2, x_3, x_4 = \left(1, \frac{4}{5}, 0, \frac{4}{5}\right)$ with $Z = 16.5$

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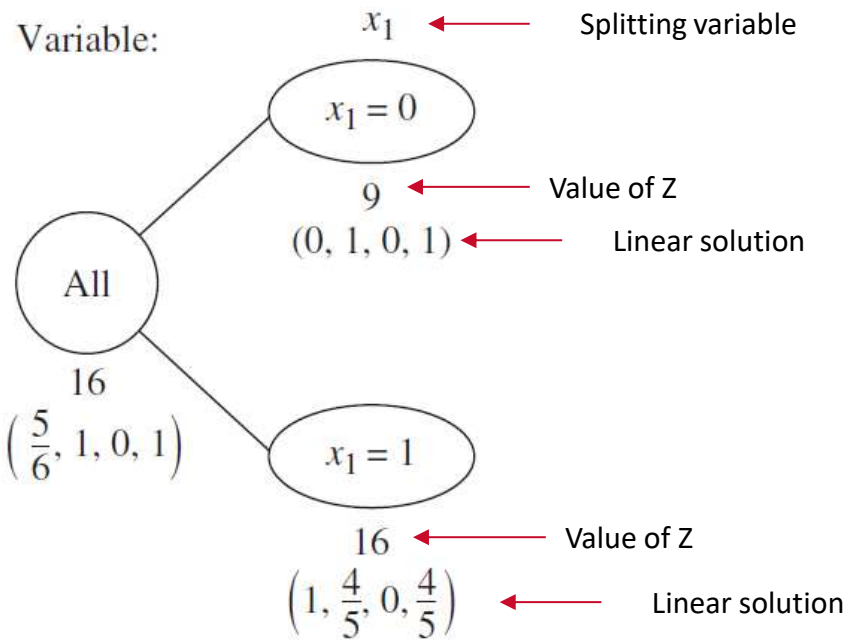
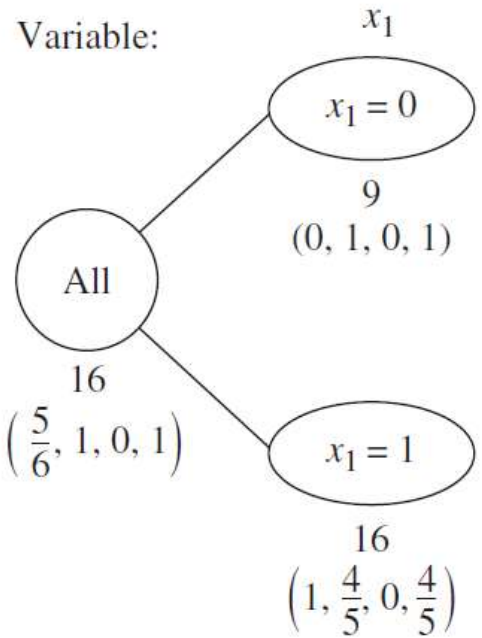
This is where we are at the end of the first bounding step:



- Fathoming (Resolving the branching at fathomed the node)

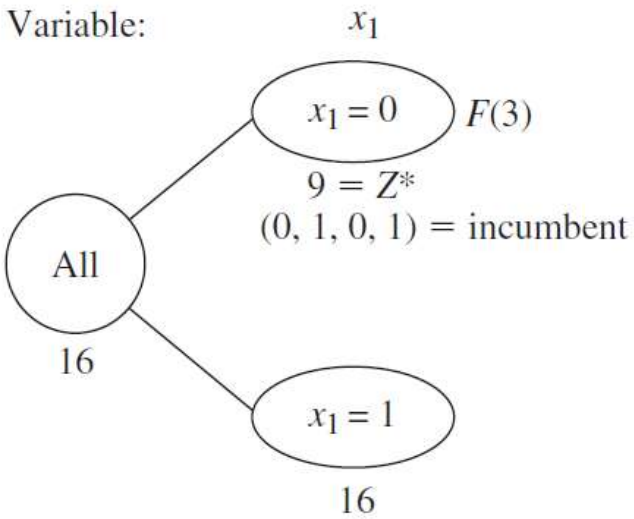


Since it is optimal it does not pay to search for other solutions in this branch

This solution is made of integers! It is **hence optimal for the subproblem with $x_1 = 0$.** We call this now **the incumbent optimum $Z^* = 9$** and say that the branch $x_1 = 0$ is fathomed; in the following we can get rid of all branches whose $Z \leq Z^* = 9$

This cannot be fathomed; we need to continue

- Fathoming (Resolving the branching at fathomed the node)



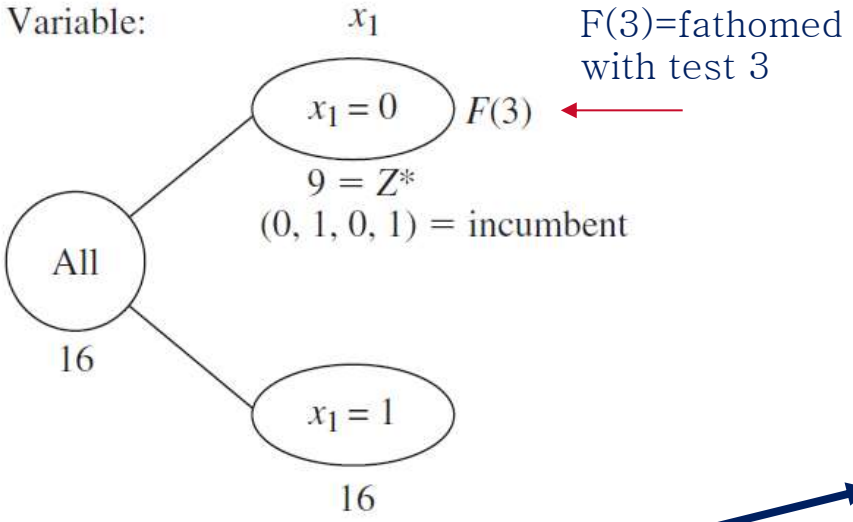
In fact, there are 3 ways of fathoming:

Test 1: Its bound by being $\leq Z^*$

Test 2: Its LP relaxation has no feasible solutions

Test 3: The optimal solution for its LP relaxation is integer.

- Fathoming (Resolving the branching at fathomed the node)



F(3)=fathomed with test 3

In fact, there are 3 ways of fathoming:

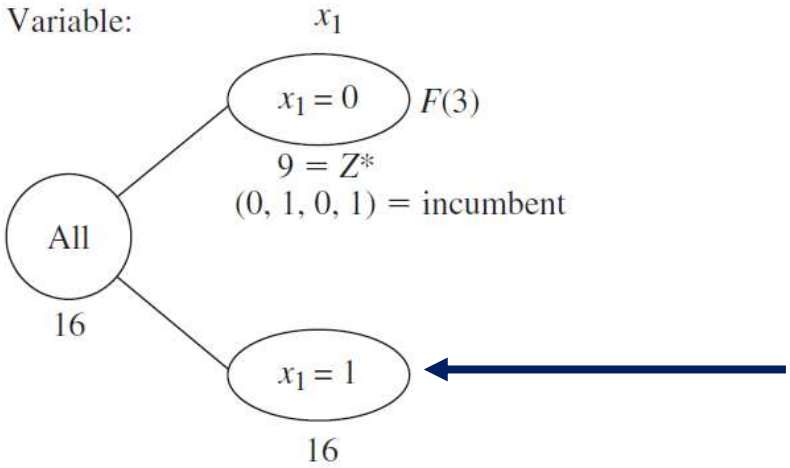
Test 1: Its bound $\leq Z^*$

Test 2: Its LP relaxation has no feasible solutions

Test 3: The optimal solution for its LP relaxation is integer

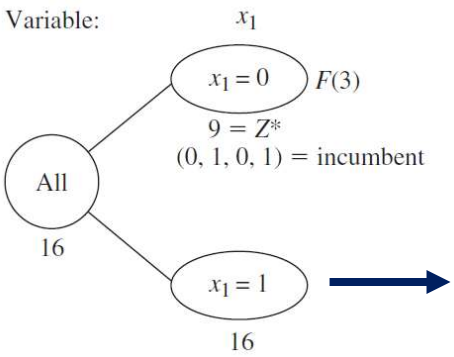
If a solution is better than the incumbent, it becomes the new incumbent Z^*, and test 1 is reapplied to all **previous unfathomed** subproblems using this new larger Z^*

- Continuing the example branching on variable x_2



We now branch the $x_1 = 1$ problem by branching x_2 between 0 and 1

- Continuing the example



(x_2 now disappears)

$x_2 = 0, x_1 = 1$

Maximize $Z = 9 + 6x_3 + 4x_4$

Subject to

Subject to:

$5x_3 + 2x_4 \leq 4$

$x_3 \leq 1$

$x_4 \leq 0$

$x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 3,4$

$x_2 = 1, x_1 = 1$

Maximize $Z = 9 + 5 + 6x_3 + 4x_4$

Subject to:

$5x_3 + 2x_4 \leq 1$

$x_3 \leq 1$

$x_4 \leq 1$

$x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 3,4$

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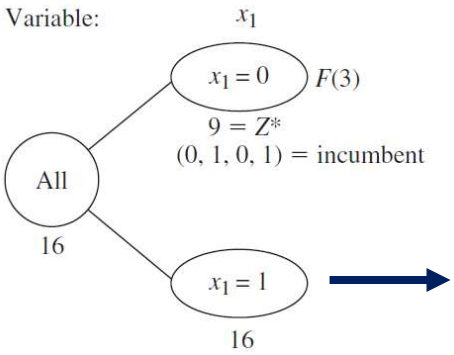
- Continuing the example



$x_2 = 0, x_1 = 1$
Maximize $Z = 9 + 6x_3 + 4x_4$
Subject to
Subject to:
$5x_3 + 2x_4 \leq 4$
$x_3 \leq 1$
$x_4 \leq 0$
$x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 3,4$

$x_2 = 1, x_1 = 1$
Maximize $Z = 9 + 5 + 6x_3 + 4x_4$
Subject to:
$5x_3 + 2x_4 \leq 1$
$x_3 \leq 1$
$x_4 \leq 1$
$x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 3,4$

Linear programming applied to these solutions yields

$x_1, x_2, x_3, x_4 = \left(1,0,\frac{4}{5},0\right)$ with $Z = 13.8$

$x_1, x_2, x_3, x_4 = \left(1,1,0,\frac{1}{2}\right)$ with $Z = 16$

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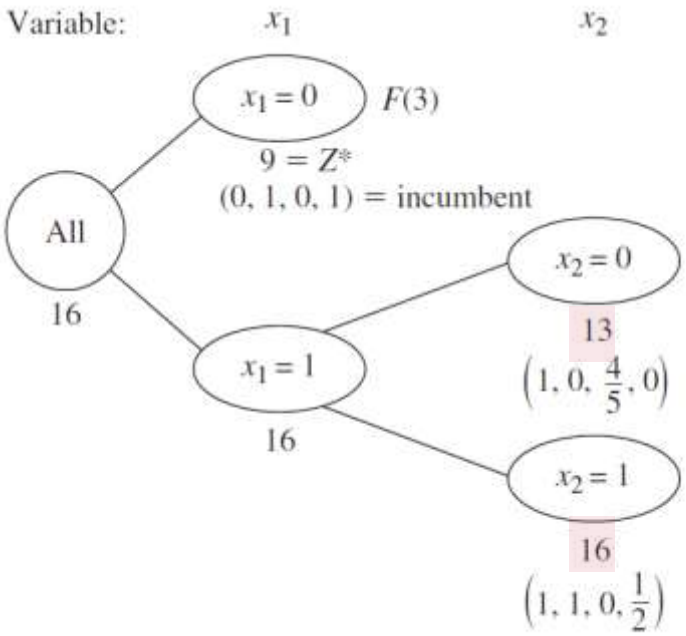
- Continuing the example



Why do I say so when they are not integer ?

Both respect constraints e.g.
$5x_3 + 2x_4 \leq 4$

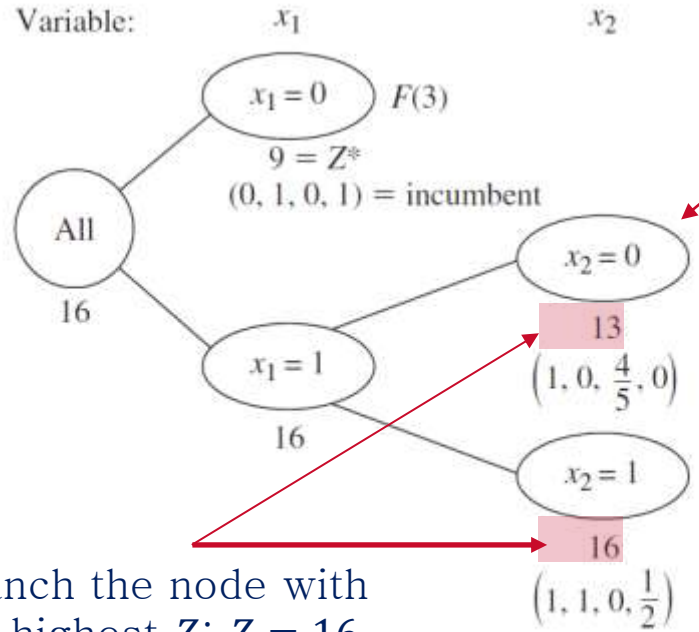
This is where we are now; no problem has been bound or fathomed at this step because

Test 1: Its bound $\leq Z^*$ No, both 13 and 16 > 9

Test 2: Its LP relaxation has no feasible solutions False; both are feasible

Test 3: The optimal solution for its LP relaxation is integer No for both

- Continuing the example

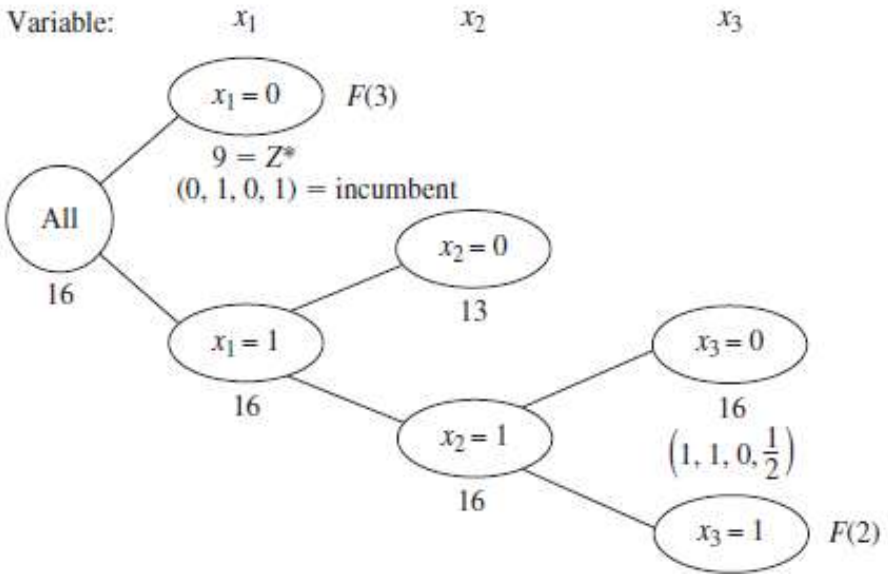


… but keep in mind that this is still open (unfathomed)

Since the problem $x_2 = 1$ has the larger Z we branch this solution splitting on x_3

Branch the node with the highest Z: $Z = 16$

- Continuing the example branching x_3 ; note how both Z and the constraints change to adopt to the new values

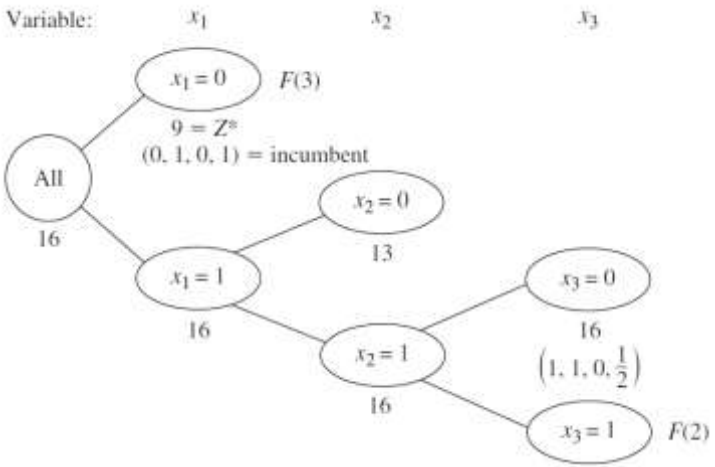


(x_3 now disappears)

$x_3 = 0, x_1 = 1, x_2 = 1$
Maximize $Z = 14 + 4x_4$
Subject to:
$2x_4 \leq 1$ This was $5x_3 + 2x_4 \leq 1$
$x_4 \leq 1$
$x_4 \leq 1$ This was $x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 4$

$x_3 = 1, x_1 = 1, x_2 = 1$
Maximize $Z = 20 + 4x_4$
Subject to:
$2x_4 \leq -4$ This was $5x_3 + 2x_4 \leq 1$
$x_4 \leq 1$
$x_4 \leq 0$ This was $x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 4$

- Continuing the example



$x_3 = 0, x_1 = 1, x_2 = 1$
Maximize $Z = 14 + 4x_4$
Subject to:
$2x_4 \leq 1$
$x_4 \leq 1$
$x_4 \leq 1$
$x_j \geq 0$ for $j = 4$

Linear programming applied to these solutions yields no feasible **integer** solution

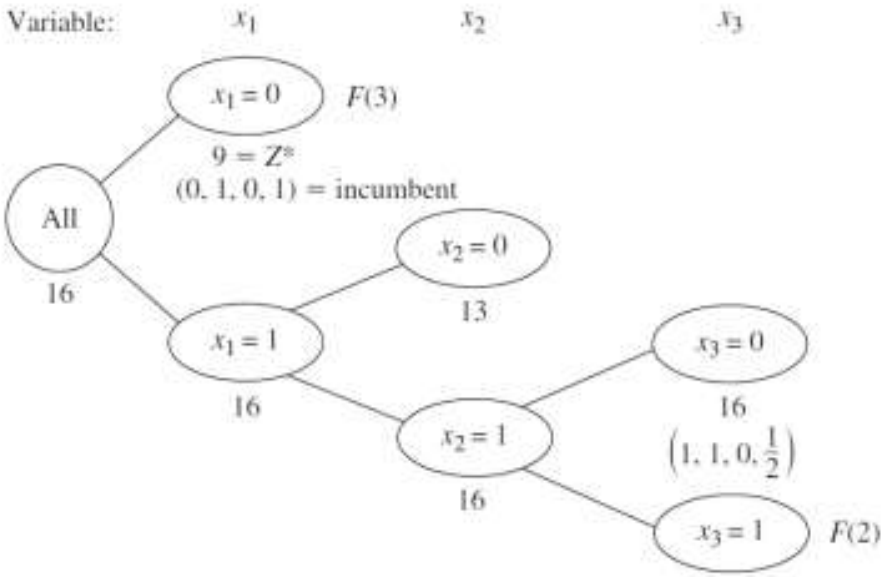
$x_1, x_2, x_3, x_4 = \left(1,1,0,\frac{1}{2}\right)$ with
$Z = 16$

$x_3 = 1, x_1 = 1, x_2 = 1$
Maximize $Z = 20 + 4x_4$
Subject to:
$2x_4 \leq -4$
$x_4 \leq 1$
$x_4 \leq 0$
$x_j \geq 0$ for $j = 4$

x_1, x_2, x_3, x_4
= no feasible solution

Why?

- Continuing the example

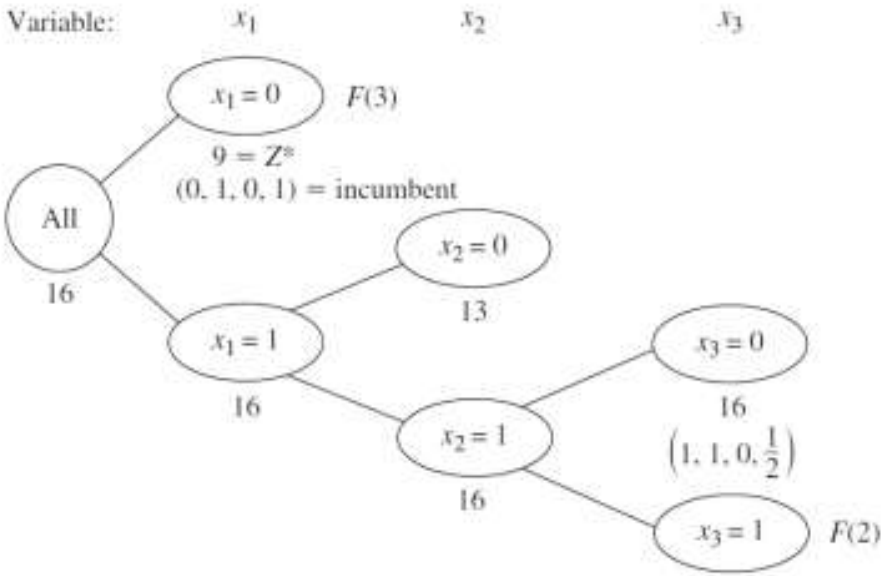


This is where we are now, with one solution ($x_3 = 1$) fathomed and one ($x_3 = 0$) open

No test failed

Test 2 failed 

- Continuing the example by branching the last variable x_4



We now branch the problem from $x_3 = 0$, but since only variable x_4 is left fixing it generates directly a solution!

For $x_4 = 0$
$x_1, x_2, x_3, x_4 = (1,1,0,0)$ with $Z = 14$

For $x_4 = 1$
$x_1, x_2, x_3, x_4 = (1,1,0,1)$ unfeasible

Why?

We can check this last passage with the original problem

For $x_4 = 0$
$x_1, x_2, x_3, x_4 = (1,1,0,0)$ with $Z = 14$

For $x_4 = 1$
$x_1, x_2, x_3, x_4 = (1,1,0,1)$ unfeasible

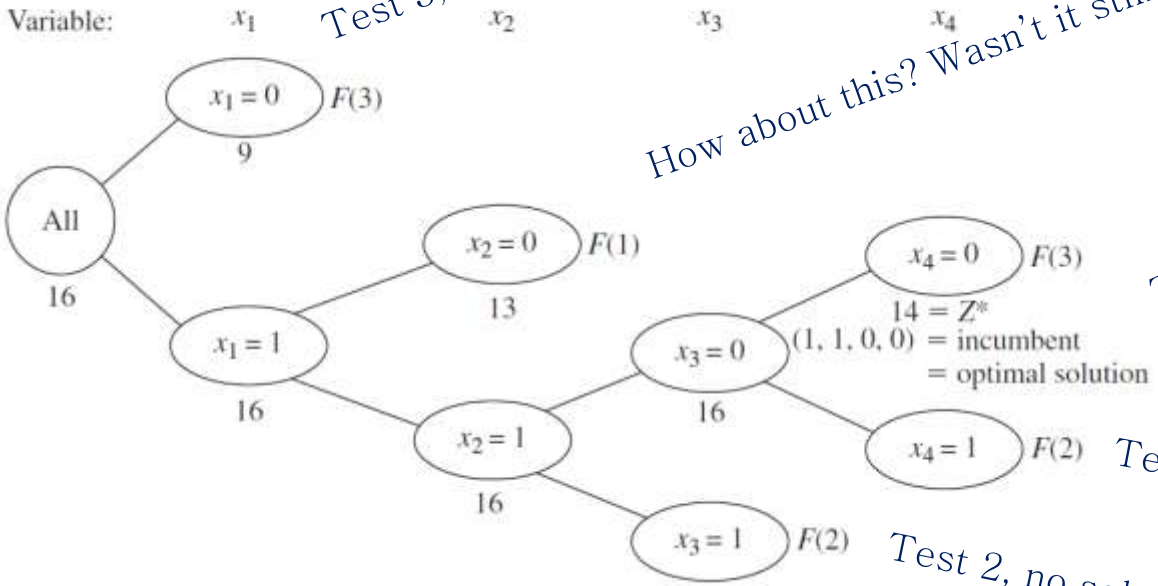
Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

$9 \leq 10$

$11 \leq 10$

- Revising and closing: the solution is $x_1, x_2, x_3, x_4 = (1,1,0,0)$ with $Z = 14$, but …



Test 3, integer solution

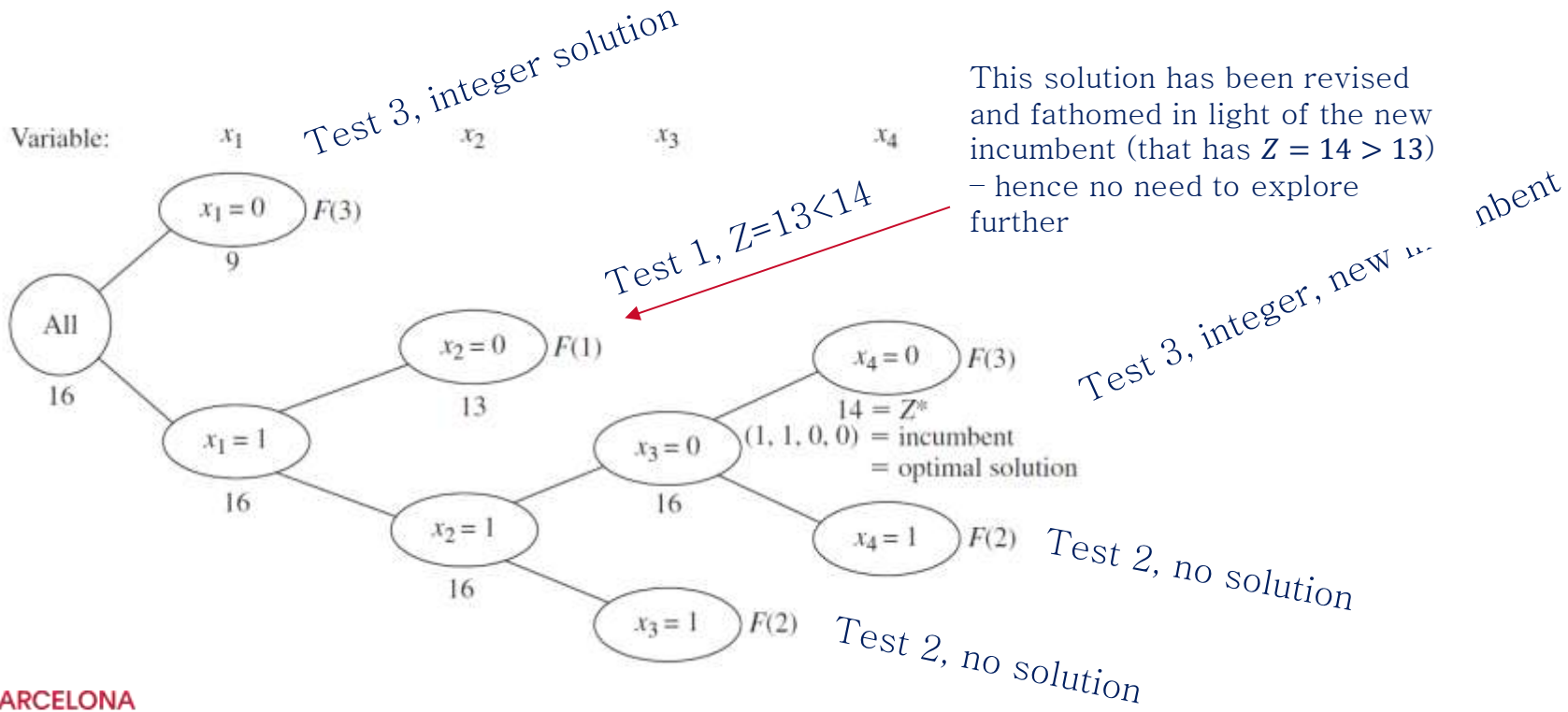
How about this? Wasn't it still open?

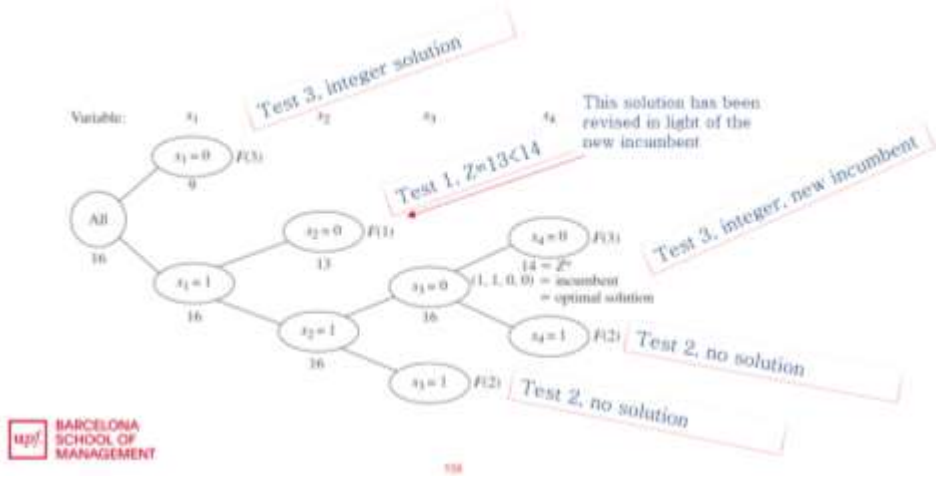
Test 3, integer, new incumbent

Test 2, no solution

Test 2, no solution

- Revising and closing: the solution is $x_1, x_2, x_3, x_4 = (1,1,0,0)$ with $Z = 14$





The solution is laborious, Needs book-keeping of how objective and constraints change in the various branches, and repeated recourse to LP, simplex calculations

For next lesson

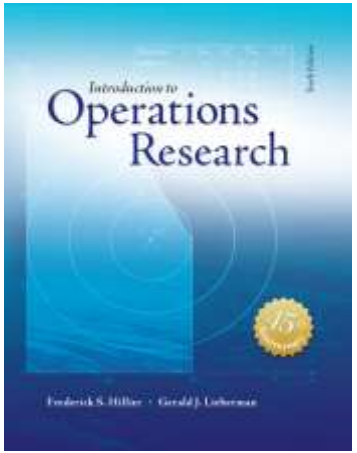
1) Install Anaconda on your laptop and bring it to class https://www.anaconda.com/download

Make sure you can use ChatGPT

Classroom exercises from

Mann, P.S. (2010) Introductory Statistics. 7th edition. Wiley.
https://www.andreasaltelli.eu/file/repository/Mann_Introductory_Statistics_7th_Ed_.pdf

Hillier and Lieberman (2015) Operation Research (10th edition)
https://www.andreasaltelli.eu/file/repository/Introduction_to_Operations_Research_10th_Frederick_S_Hillier.pdf



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1) Read pages 208-225 of the Mann book and solve all exercises from 5.41 to 5.44. It is not forbidden to use Excel.

5.41 A ski patrol unit has nine members available for duty, and two of them are to be sent to rescue an injured skier. In how many ways can two of these nine members be selected? Now suppose the order of selection is important. How many arrangements are possible in this case?

5.42 An ice cream shop offers 25 flavors of ice cream. How many ways are there to select 2 different flavors from these 25 flavors? How many permutations are possible?

5.43 A veterinarian assigned to a racetrack has received a tip that one or more of the 12 horses in the third race have been doped. She has time to test only 3 horses. How many ways are there to randomly select 3 horses from these 12 horses? How many permutations are possible?

5.44 An environmental agency will randomly select 4 houses from a block containing 25 houses for a radon check. How many total selections are possible? How many permutations are possible?

Classroom (2)

5.46 A company employs a total of 16 workers. The management has asked these employees to select 2 workers who will negotiate a new contract with management. The employees have decided to select the 2 workers randomly. How many total selections are possible? Considering that the order of selection is important, find the number of permutations.

Classroom

3) Solve Hillier online book problem 12.1.3 page 534, only question (a) Formulate a BIP model for this problem.

12.1-3. A real estate development firm, Peterson and Johnson, is considering five possible development projects. The following table shows the estimated long-run profit (net present value) that each project would generate, as well as the amount of investment required to undertake the project, in units of millions of dollars. The owners of the firm, Dave Peterson and Ron Johnson, have raised $20 million of investment capital for these projects. Dave and Ron now want to select the combination of projects that will maximize their total estimated long-run profit (net present value) without investing more that $20 million. (a) Formulate a BIP model for this problem.

	Development Project				
	1	2	3	4	5
Estimated profit	1	1.8	1.6	0.8	1.4
Capital required	6	12	10	4	8

Classroom

4) Solve Hillier online book problem 12.1.4 page 534, only question (a) Formulate a BIP model for this problem.

2.1-4. The board of directors of General Wheels Co. is considering six large capital investments. Each investment can be made only once. These investments differ in the estimated long-run profit (net present value) that they will generate as well as in the amount of capital required, as shown by the following table (in units of millions of dollars):

	Investment Opportunity					
	1	2	3	4	5	6
Estimated profit	15	12	16	18	9	11
Capital required	38	33	39	45	23	27

The total amount of capital available for these investments is $100 million. Investment opportunities 1 and 2 are mutually exclusive, and so are 3 and 4. Furthermore, neither 3 nor 4 can be undertaken unless one of the first two opportunities is undertaken. There are no such restrictions on investment opportunities 5 and 6. The objective is to select the combination of capital investments that will maximize the total estimated long-run profit (net present value). (a) Formulate a BIP model for this problem.

5) Solve Hillier online book problem 12.3-1. page 535, only question (a) Formulate a BIP model for this problem.

2.3-1.* The Research and Development Division of the Progressive Company has been developing four possible new product lines. Management must now make a decision as to which of these four products actually will be produced and at what levels. Therefore, an operations research study has been requested to find the most profitable product mix. A substantial cost is associated with beginning the production of any product, as given in the first row of the following table. Management's objective is to find the product mix that maximizes the total profit (total net revenue minus start-up costs). Let the continuous decision variables x1, x2, x3, and x4 be the production levels of products 1, 2, 3, and 4, respectively. Management has imposed the following policy constraints on these variables:

1. No more than two of the products can be produced.

2. Either product 3 or 4 can be produced only if either product 1 or 2 is produced.

3. Either

$$5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6{,}000$$

Or

$$4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6{,}000$$

	Product			
	1	2	3	4
Start-up cost	\$50,000	\$40,000	\$70,000	\$60,000
Marginal revenue	\$70	\$60	\$90	\$80

(a) Introduce auxiliary binary variables to formulate a mixed BIP model for this problem.

Thank you

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