

Máster Universitario en Administración y Dirección de Empresas Full Time MBA

Quantitative methods for decision making

Professor Andrea Saltelli

Where to find this talk

August 25 2023: The politics of modelling is out!



Praise for the volume

"A long-awaited examination of the role—and obligation—of modeling."

Nassim Nicholas Taleb, Distinguished Professor of Risk Engineering, NYU Tandon School of Engineering. Author, of the 5-volume series *Incerto*.

"A breath of fresh air and a much needed cautionary view of the ever-widening dependence on mathematical modeling."

Orrin H. Pilkey, Professor at Duke University's Nicholas School of the Environment, co-author with Linda Pilkey-Jarvis of *Useless Arithmetic: Why Environmental Scientists Can't Predict the Future*, Columbia University Press 2009.

Mastodon Toots by

@AndreaSaltelli



AndreaSaltelli

2023/10/11 11:03

Thanks to Maria Kozlova of LUT University in Finland for taking and curating this recording. My trajectory from number crunching to thinking about numbers' role in human affairs

[youtube.com/watch?v=wv0C-PwE11k](https://www.youtube.com/watch?v=wv0C-PwE11k)

View on [mastodon.social](#)

The talk is also at

<https://ecampus.bsm.upf.edu/>,

where you find additional reading material

Elements of quantification for decision making with emphasis on operation research

Midterm

- 1) Disease A affects one person in 100,000 in a given population and a test has a rate of false positive of 5%. If B denotes the event of having the test positive and using the formula of Bayes $P(A|B)P(B) = P(B|A)P(A)$ which is the probability of a person having the disease if she tests positive? Assume $P(B|A)=1$, meaning that there are no false negative: all people with disease A test positive.

1) Solution:

Over 100,000 people
the test will be positive
for 5000+1

$$P(A) = .00001$$

$$P(B) \approx .05$$

$$P(B|A) = 1$$

$$P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$$

$$\Rightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{.00001}{.05} = 0.0002 = .02\%$$

2) List the name of the four main assumptions of linear programming

Solution:

Proportionality

Divisibility

Additivity

Certainty

3) Which of the following constraints ~~is~~ **are not** linear

a) $9x_1 + 3x_2x_3 \leq 6$

b) $4x_1 + 2x_2^3 \geq 9$

c) $x_1 + 3x_2 \leq 6$

d) $5\sqrt{x_1} + 3x_2 + 2x_3 \leq 25$

Solution: a, b, d

4) Launching two coins 6 times which is the probability of getting two heads exactly twice.

$$\text{Solution: } \binom{6}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{6-2} = 15 \frac{3^4}{4^6} = .297$$

Experiment: launching 2 coins

$$\binom{6}{2} = \frac{6!}{2! * 4!} = \frac{6 * 5}{2} = 15$$

Space of the events {HH,TT,HT,TH}

Event being sought HH; $P(H)=1/4$

Number of replications of the experiment 6

Occurrences of the event 2

5) For each of the following constraints, draw a separate graph to show the nonnegative solutions that satisfy this constraint.

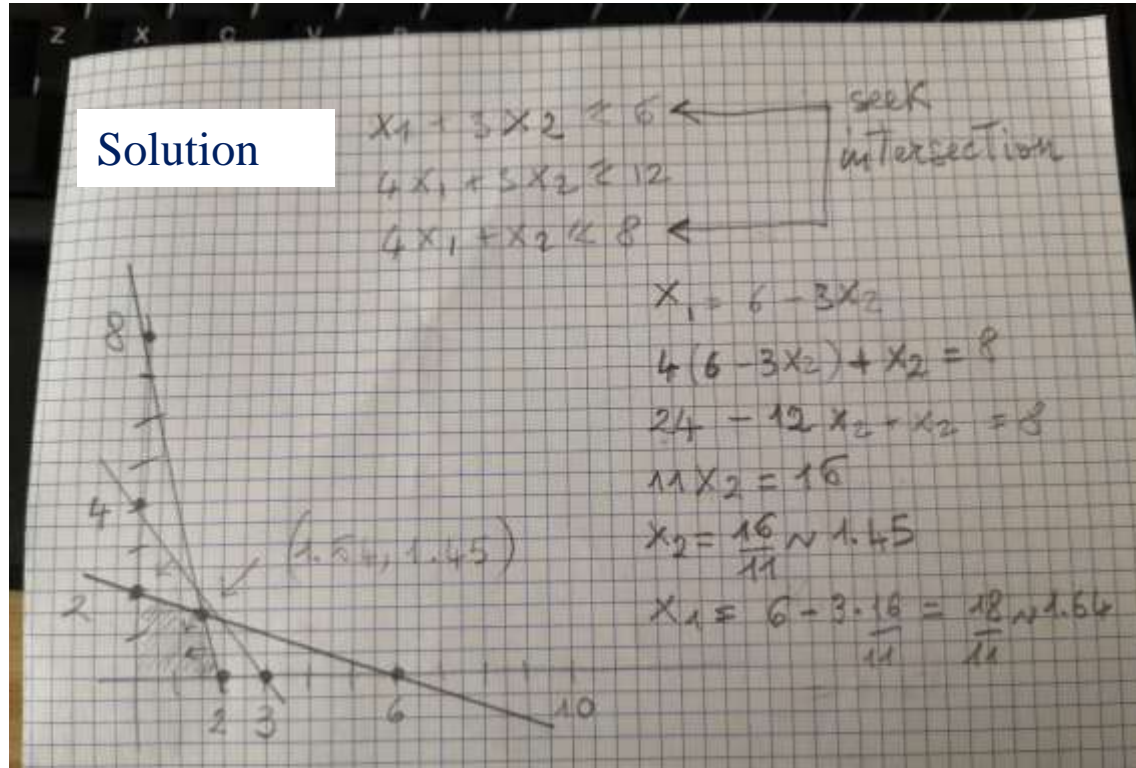
$$x_1 + 3x_2 \leq 6$$

$$4x_1 + 3x_2 \leq 12$$

$$4x_1 + x_2 \leq 8$$

Now combine these constraints into a single graph to show the feasible region for the entire set of functional constraints plus nonnegativity constraints.

Solution



6) The Medequip Company produces precision medical diagnostic equipment at two factories. Three medical centers have placed orders for this month's production output. The table below shows what the cost would be for shipping each unit from each factory to each of these customers. Also shown are the number of units that will be produced at each factory and the number of units ordered by each customer.

From \ To	Unit Shipping Cost			Output
	Customer 1	Customer 2	Customer 3	
Factory 1	\$600	\$800	\$700	400 units
Factory 2	\$400	\$900	\$600	500 units
Order size	300 units	200 units	400 units	

A decision now needs to be made about the shipping plan for how many units to ship from each factory to each customer.

(a) Formulate a linear programming model for this problem (without solving it).

(b) Knowing that the solution is

From Factory 1, ship 200 units to Customer 2 and 200 units to Customer 3.

From Factory 2, ship 300 units to Customer 1 and 200 units to Customer 3.

Check that this solution satisfies the constraints you have written.

Solution: Let x_{ij} be the cost of shipping from factory i to customer j , then

Minimize

$$600x_{11} + 800x_{12} + 700x_{13} + 400x_{21} + 900x_{22} + 600x_{23}$$

Subject to

$$x_{11} + x_{12} + x_{13} = 400$$

$$x_{21} + x_{22} + x_{23} = 500$$

$$x_{11} + x_{21} = 300$$

$$x_{12} + x_{22} = 200$$

$$x_{13} + x_{23} = 400$$

$$\text{and } x_{ij} \geq 0 \text{ for all } i, j$$

Given solutions

$$x_{11}, x_{12}, x_{13} = 0, 200, 200$$

$$x_{21}, x_{22}, x_{23} = 300, 0, 200$$

$$Z = 800 \cdot 200 + 700 \cdot 200 + 400 \cdot 300 + 600 \cdot 200 = 10,000 \cdot (16 + 14 + 12 + 12) = 540,000$$

Verify:

$$0 + 200 + 200 = 400$$

$$300 + 0 + 200 = 500$$

$$0 + 300 = 300$$

$$200 + 0 = 200$$

$$200 + 200 = 400$$

7) In a sample survey, 1,800 senior citizens were asked whether or not they have ever been victimized by a dishonest telemarketer. The following table gives the responses by age group.

Suppose one person is randomly selected from these senior citizens. Find the following probabilities.

a) $P(\text{have been victimized or } B)$

b) $P(\text{have never been victimized or } C)$

Launching two coins 6 times which is the probability of getting two heads exactly twice.

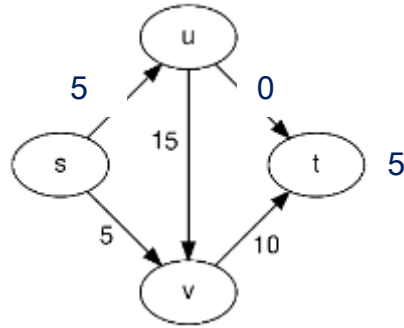
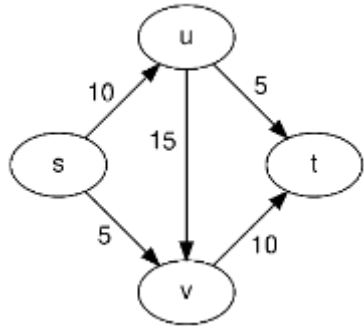
			Have Been Victimized	Have Never Been Victimized	
Age	60–69	(A)	106	698	804
	70–79	(B)	145	447	592
	80 or over	(C)	61	343	404
			312	1488	1800

7) Solution

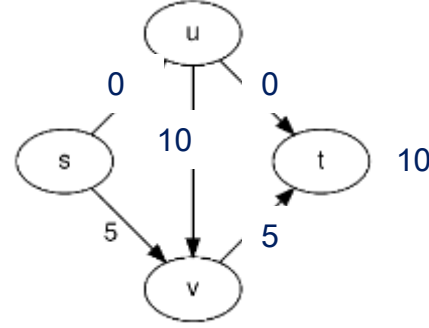
$$P(V \cup B) = P(V) + P(B) - P(V \cap B) = \frac{312 + 592 - 145}{1800} = 0.42$$

$$P(NV \cup C) = P(NV) + P(C) - P(NV \cap C) = \frac{1488 + 404 - 343}{1800} = 0.86$$

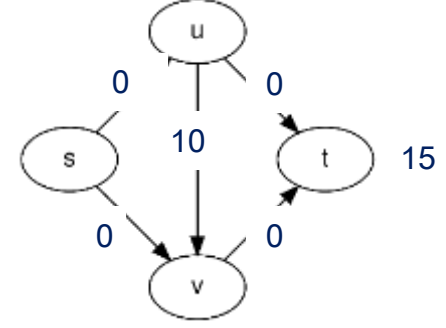
8) Solve the **maximum flow** problem from node s to node t for this network showing your steps. Numbers on the links are **capacities**.



Move 5 via $s \rightarrow u \rightarrow t$



Move 5 via $s \rightarrow u \rightarrow v \rightarrow t$



Move 5 via $s \rightarrow v \rightarrow t$

Solution=15 units

Homework lesson #5

Please use these versions of the textbooks

Mann, P.S. (2010) Introductory Statistics. 7th edition.
Wiley.

https://www.andreasaltelli.eu/file/repository/Mann_Introductory_Statistics_7th_Ed_.pdf



Hillier and Lieberman (2015) Operation Research (10th edition)

https://www.andreasaltelli.eu/file/repository/Introduction_to_Operations_Research_10th_Frederick_S_Hillier.pdf



2) Read pages 208–225 of the Mann book and solve all exercises from 5.41 to 5.44. It is not forbidden to use Excel.

5.41 A ski patrol unit has nine members available for duty, and two of them are to be sent to rescue an injured skier. In how many ways can two of these nine members be selected? Now suppose the order of selection is important. How many arrangements are possible in this case?

Combinations $\binom{9}{2} = \frac{9!}{2!7!} = \frac{9 \cdot 8}{2} = 36$

Permutations $\frac{9!}{(9-2)!} = 9 \cdot 8 = 72$

5.42 An ice cream shop offers 25 flavors of ice cream. How many ways are there to select 2 different flavors from these 25 flavors? How many permutations are possible?

Combinations $\binom{25}{2} = \frac{25!}{2!23!} = \frac{25 \cdot 24}{2} = 300$

Permutations $\frac{25!}{(25-2)!} = 25 \cdot 24 = 600$



2) Read pages 208–225 of the Mann book and solve all exercises from 5.41 to 5.44. It is not forbidden to use Excel.

5.43 A veterinarian assigned to a racetrack has received a tip that one or more of the 12 horses in the third race have been doped. She has time to test only 3 horses. How many ways are there to randomly select 3 horses from these 12 horses? How many permutations are possible?

$$\text{Combinations } \binom{12}{3} = \frac{12!}{3!9!} = \frac{12*11*10}{3*2} = 220$$

$$\text{Permutations } \frac{12!}{(12-3)!} = 12 * 11 * 10 = 1,320$$

5.44 An environmental agency will randomly select 4 houses from a block containing 25 houses for a radon check. How many total selections are possible? How many permutations are possible?

$$\text{Combinations } \binom{25}{4} = \frac{25!}{4!21!} = \frac{25*24*23*22}{4*3*2} = 12,650$$

$$\text{Permutations } \frac{25!}{(25-4)!} = 25 * 24 * 23 * 22 = 303,600$$



In class:

5.46 A company employs a total of 16 workers. The management has asked these employees to select 2 workers who will negotiate a new contract with management. The employees have decided to select the 2 workers randomly. How many total selections are possible? Considering that the order of selection is important, find the number of permutations.

A) 120

B) 240



Homework

3) Solve Hillier online book problem 12.1.3 page 534, only question (a) Formulate a BIP model for this problem.

12.1-3. A real estate development firm, Peterson and Johnson, is considering five possible development projects. The following table shows the estimated long-run profit (net present value) that each project would generate, as well as the amount of investment required to undertake the project, in units of millions of dollars. The owners of the firm, Dave Peterson and Ron Johnson, have raised \$20 million of investment capital for these projects. Dave and Ron now want to select the combination of projects that will maximize their total estimated long-run profit (net present value) without investing more that \$20 million. (a) Formulate a BIP model for this problem.

	Development Project				
	1	2	3	4	5
Estimated profit	1	1.8	1.6	0.8	1.4
Capital required	6	12	10	4	8

Maximize

$$Z = x_1 + 1.8x_2 + 1.6x_3 + 0.8x_4 + 1.4x_5$$

Subject to

$$6x_1 + 12x_2 + 10x_3 + 4x_4 + 8x_5 \leq 20$$

and

$$x_j, j = 1, 2, \dots, 6 \text{ binary}$$

Homework

4) Solve Hillier online book problem 12.1.4 page 534, only question (a) Formulate a BIP model for this problem.

2.1–4. The board of directors of General Wheels Co. is considering six large capital investments. Each investment can be made only once. These investments differ in the estimated long-run profit (net present value) that they will generate as well as in the amount of capital required, as shown by the following table (in units of millions of dollars):

	Investment Opportunity					
	1	2	3	4	5	6
Estimated profit	15	12	16	18	9	11
Capital required	38	33	39	45	23	27

The total amount of capital available for these investments is \$100 million. Investment opportunities 1 and 2 are mutually exclusive, and so are 3 and 4. Furthermore, neither 3 nor 4 can be undertaken unless one of the first two opportunities is undertaken. There are no such restrictions on investment opportunities 5 and 6. The objective is to select the combination of capital investments that will maximize the total estimated long-run profit (net present value). (a) Formulate a BIP model for this problem.

4) Solve Hillier online book problem 12.1.4 page 534, only question (a) Formulate a BIP model for this problem.

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Maximize

$$Z = 15x_1 + 12x_2 + 16x_3 + 18x_4 + 9x_5 + 11x_6$$

Subject to

$$38x_1 + 33x_2 + 39x_3 + 45x_4 + 23x_5 + 27x_6 \leq 100$$

... and subject to

$$x_1 + x_2 \leq 1$$

$$x_3 + x_4 \leq 1$$

$$x_3 \leq x_1 + x_2$$

$$x_4 \leq x_1 + x_2$$

with $x_j, j = 1, 2, \dots, 6$ binary

5) Solve Hillier online book problem 12.3-1. page 535, only question (a) Formulate a BIP model for this problem.

2.3-1.* The Research and Development Division of the Progressive Company has been developing four possible new product lines. Management must now make a decision as to which of these four products actually will be produced and at what levels. Therefore, an operations research study has been requested to find the most profitable product mix. A substantial cost is associated with beginning the production of any product, as given in the first row of the following table. Management's objective is to find the product mix that maximizes the total profit (total net revenue minus start-up costs). Let the continuous decision variables x_1 , x_2 , x_3 , and x_4 be the production levels of products 1, 2, 3, and 4, respectively. Management has imposed the following policy constraints on these variables:

1. No more than two of the products can be produced.

2. Either product 3 or 4 can be produced only if either product 1 or 2 is produced.

3. Either

$$5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6,000$$

Or

$$4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6,000$$

	Product			
	1	2	3	4
Start-up cost	\$50,000	\$40,000	\$70,000	\$60,000
Marginal revenue	\$70	\$60	\$90	\$80

(a) Introduce auxiliary binary variables to formulate a mixed BIP model for this problem.

5) Solve Hillier online book problem 12.3-1, page 535, only question (a) Formulate a BIP model for this problem.

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Start-up cost	\$50,000	\$40,000	\$70,000	\$60,000
Marginal revenue	\$70	\$60	\$90	\$80

(a) Introduce auxiliary binary variables to formulate a mixed BIP model for this problem.

Maximize

$$(70x_1 - 50,000)y_1 + (60x_2 - 40,000)y_2 + (90x_3 - 70,000)y_3 + (80x_4 - 60,000)y_4$$

Subject to:

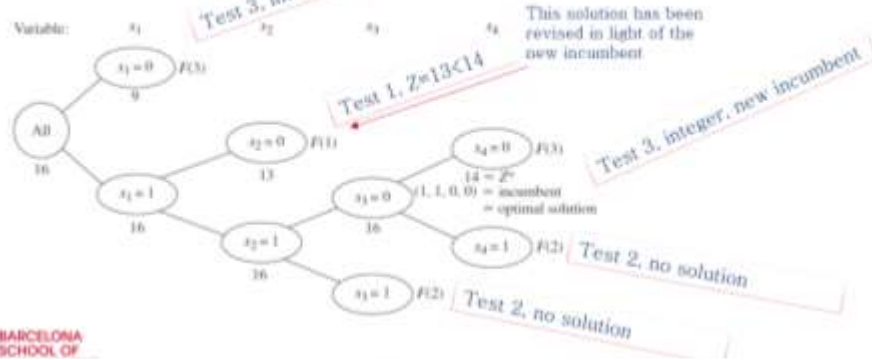
$$y_3 \leq y_1 + y_2 \quad 5x_1 + 3x_2 + 6x_3 + 4x_4 \leq 6,000 + My_5$$

$$y_1 + y_2 + y_3 + y_4 \leq 2$$

$$y_4 \leq y_1 + y_2$$

$$4x_1 + 6x_2 + 3x_3 + 5x_4 \leq 6,000 + M(1 - y_5)$$

Where were we left?



■ TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million

Capital available: \$10 million

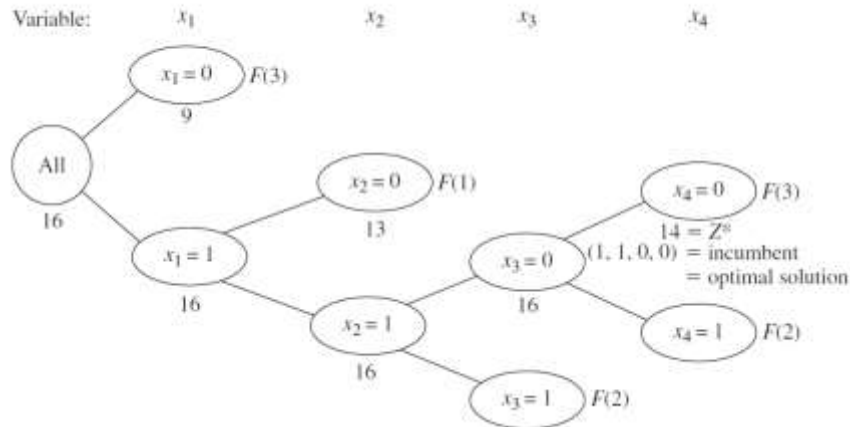
The solution was laborious, Needs book-keeping of how objectives and constraints change in the various branches, and repeated recourse to LP, simplex calculations



The choice was if building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building **at most one** new warehouse, but the choice of location is restricted to a city where a new factory is being built.

But this was a very simple problem in 4 variables;

- Would it have been possible to solve this problem 'brute force'?
- How many leaves would there be in this tree in total?
- Would we need to apply LP?
- Do you agree that each string of 4 zero or one number (example 0110) describes a leaf?
- Please write them down



0000 0
0001 1
0010 2
0011 3
0100 4
0101 5
0110 6
0111 7
1000 8
1001 9
1010 10
1011 11
1100 12
1101 13
1110 14
1111 15



Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$

Subject to:

$$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

$$-x_1 + x_3 \leq 0$$

$$-x_2 + x_4 \leq 0$$

$$x_3 + x_4 \leq 1$$

and

x_j binary for $j = 1, 2, 3, 4$

In this set of slides:

16 Nonlinear programming

17 Decision Analysis

16.

Nonlinear Programming

Problem framing and examples. Graphical illustration. Bisection and Newton methods of solution. Metaheuristics. Genetic algorithms. Examples of nonlinear models. Python coding. Exploration versus optimization. Hillier 2014, chapters 13 and 14.

Problem setting

Finding values of $\mathbf{x} = (x_1, x_2, \dots, x_n)$ as to maximize or minimize a generic function $f(\mathbf{x})$ subject to

← Is this specification necessary?

$$g_i(\mathbf{x}) \leq b_i \quad i = 1, 2, \dots, m$$

and

$$\mathbf{x} \geq 0$$

Problem setting

Finding values of $\mathbf{x} = (x_1, x_2, \dots, x_n)$ as to maximize or minimize a generic function $f(\mathbf{x})$ subject to

$$g_i(\mathbf{x}) \leq b_i \quad i = 1, 2, \dots, m$$

and

$$\mathbf{x} \geq 0$$

3) Which of the following constraints is **are not** linear

a) $9x_1 + 3x_2x_3 \leq 6$

b) $4x_1 + 2x_2^3 \geq 9$

c) $x_1 + 3x_2 \leq 6$

d) $5\sqrt{x_1} + 3x_2 + 2x_3 \leq 25$



Remember the midterm test

A Standard Form of the Model:

Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

And to:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \dots, \quad x_n \geq 0.$$

Z = value of overall measure of performance

x_j = decision variables, level of activity j for $j = 1, 2, \dots, n$

a_j^i = amount of resource i consumed by each unit of activity j

b_i amount of resource i that is available for allocation to activities $i = 1, 2, \dots, m$

c_j increase in Z that would result from each unit increase in level of activity

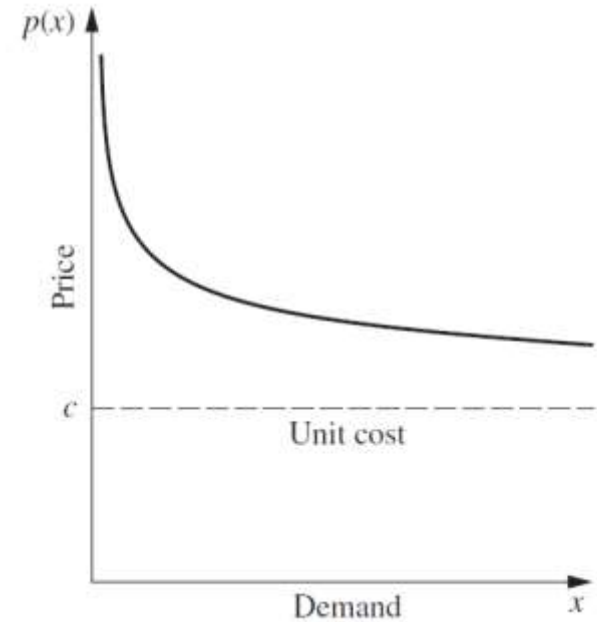


What is the difference from the linear problem?

Many ways in which a linear problem can become nonlinear

Linear: there is a fixed unit profit associated with each product, so the resulting objective function will be linear

Nonlinear: prices $p(x)$ are subject to elasticity



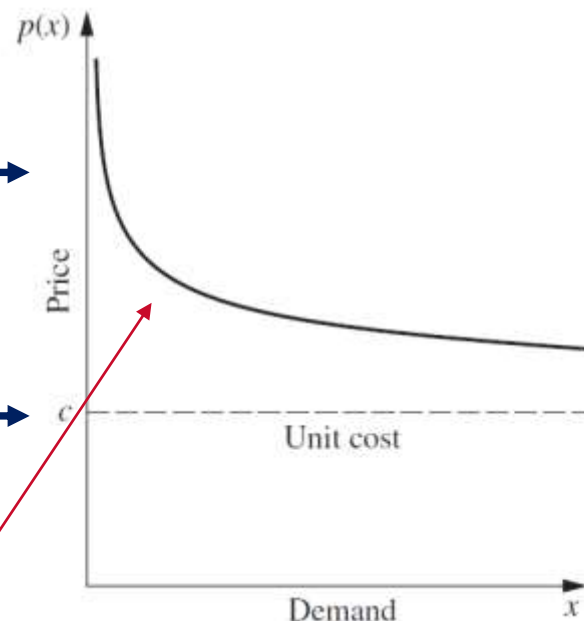
The firm's profit P from producing and selling x units is given by the nonlinear function

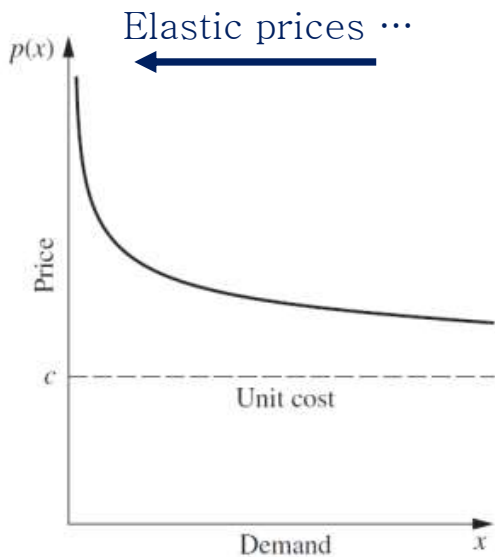
$$P(x) = xp(x) - cx$$

But the higher the price,
the less is sold ... →

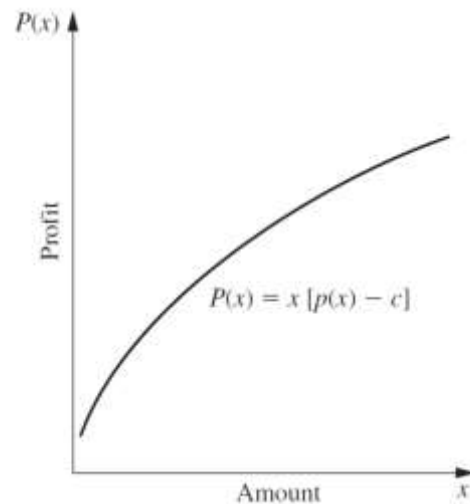
... While here the
production cost →
remain constant

How should this curve
change to make the
problem linear?





... Lead to non linear profits



The firm's profit P from producing and selling x units is given by the nonlinear function

$$P(x) = xp(x) - cx$$

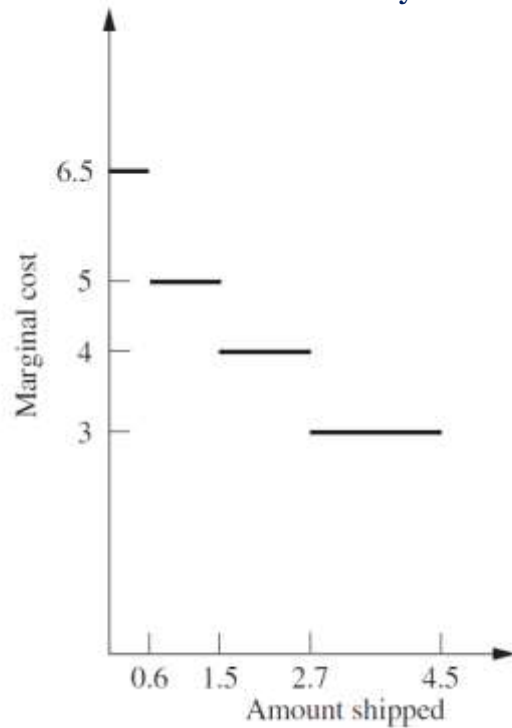
But the production cost may as well be a non linear function, e.g. in the case of increasing or diminishing returns

$$P(x) = xp(x) - cx$$



What happens to c in the two cases?
Is c itself a function of x ?

In transportation problems the shipping costs may decrease with the volume shipped

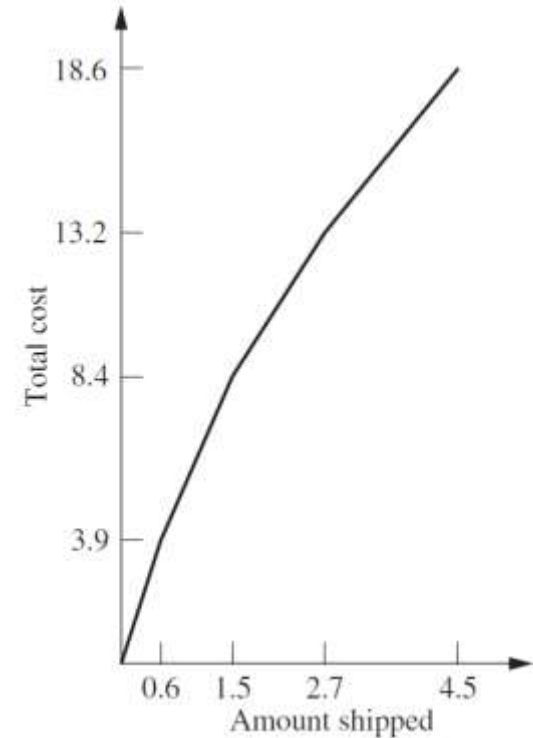


Decreasing unitary cost

...



... Piecewise total cost



In portfolio modelling the decision variable could be the number of shares of a given stock $x_j, i = 1, 2, \dots, n$ to be included

Assume known the mean return of stock μ_j and its variance σ_{jj} . Also assume that σ_{jj} is a proxy of the risk for that stock.

But the fluctuations of the stock are not independent, so we also need to know the covariances $\sigma_{ij}, i \neq j, i, j = 1, 2, \dots, n$

Return from the entire portfolio $R(\mathbf{x}) = \sum_j^n \mu_j x_j$

Variance $V(\mathbf{x})$ of the total return $V(\mathbf{x}) = \sum_{j=1}^n \sum_{i=1}^n \sigma_{ij} x_i x_j$



Source: <https://www.britannica.com/money/topic/stock-exchange-finance>

So the nonlinear problem is
Minimize

$$V(\mathbf{x}) = \sum_{j=1}^n \sum_{i=1}^n \sigma_{ij} x_i x_j$$

But is this
correct?

... or should it be

$$V(\mathbf{x}) = \sum_{j=1}^n \sum_{i=1}^j \sigma_{ij} x_i x_j$$

Subject to

$\sum_j^n \mu_j x_j \geq \mathbf{L}$ where \mathbf{L} is the minimum profit desired

$\sum_j^n P_j x_j \leq \mathbf{B}$ where P_j is the cost of stock j and \mathbf{B} is the budget available for the portfolio

$x_j \geq 0, i = 1, 2, \dots, n$
(Hiller, p. 551)



Source: <https://www.britannica.com/money/topic/stock-exchange-finance>

$$V(\mathbf{x}) = \sum_{j=1}^n \sum_{i=1}^j \sigma_{ij} x_i x_j$$

So the nonlinear problem is
Minimize

$$V(\mathbf{x}) = \sum_{j=1}^n \sum_{i=1}^n \sigma_{ij} x_i x_j$$

Subject to

$\sum_j^n \mu_j x_j \geq \mathbf{L}$ where \mathbf{L} is the minimum profit desired

$\sum_j^n P_j x_j \leq \mathbf{B}$ where P_j is the cost of stock j and \mathbf{B} is the budget available for the portfolio

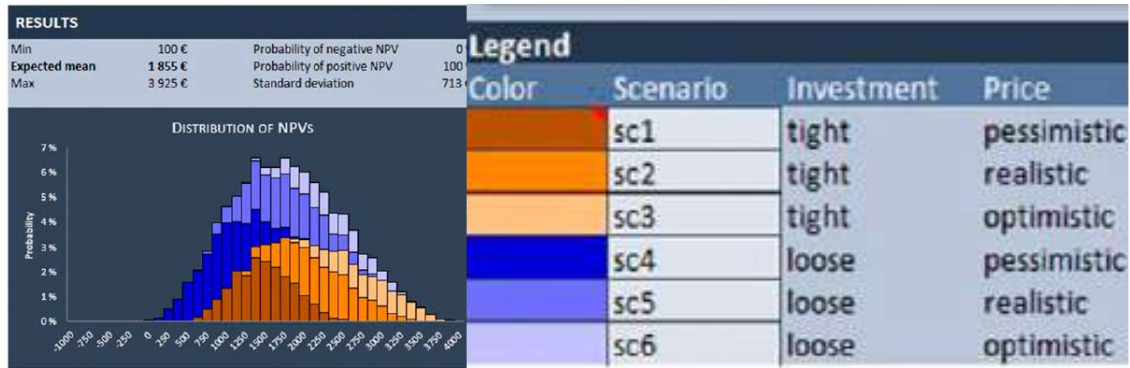
$x_j \geq 0, i = 1, 2, \dots, n$

There is a trade off between \mathbf{L} and $V(\mathbf{x})$

(higher profit associated to higher risk)

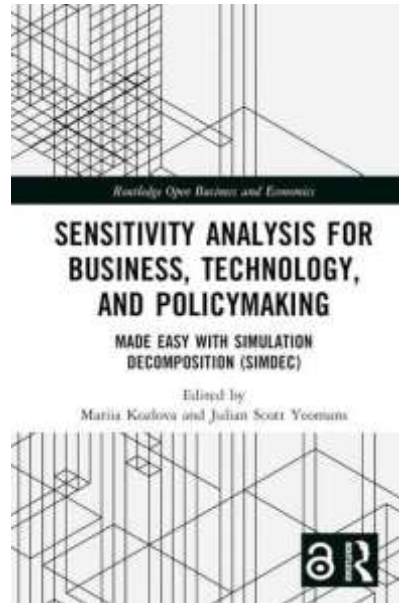
So that this problem is solved for a range of values of \mathbf{L} and comparing for each \mathbf{L} the associated $R(\mathbf{x}) = \sum_j^n \mu_j x_j$ and $V(\mathbf{x})$

“Therefore, rather than stopping with one choice of L , it is common to use a *parametric* (nonlinear) programming approach to generate the optimal solution as a function of L over a wide range of values of L . The next step is to examine the values of $R(\mathbf{x})$ and $V(\mathbf{x})$ for these solutions that are optimal for some value of L and then to choose the solution that seems to give the best trade-off between these two quantities.” (Hillier, p. 552)



Agree: better to look at the entire output distribution of the return $R(\mathbf{x}) = \sum_j^n \mu_j x_j$, including its tails, and have a sensitivity analysis to go with it, as to learn where the worst outcomes are coming from

From Lesson #3



“Therefore, rather than stopping with one choice of L , it is common to use a *parametric* (nonlinear) programming approach to generate the optimal solution as a function of L over a wide range of values of L . The next step is to examine the values of $R(\mathbf{x})$ and $V(\mathbf{x})$ for these solutions that are optimal for some value of L and then to choose the solution that seems to give the best trade-off between these two quantities.” (Hillier, p. 552)

The estimation of the σ_{ij} is a delicate matter;
the case of the subprime mortgage crisis

Nassim Nicholas Taleb, hedge fund manager and author of *The Black Swan*, is particularly harsh when it comes to the copula. "People got very excited about the Gaussian copula because of its mathematical elegance, but the thing never worked," he says. "Co-association between securities is not measurable using correlation," because past history can never prepare you for that one day when everything goes south. "Anything that relies on correlation is charlatanism."

[« Back to Article](#)

WIRED MAGAZINE: 17.03

Recipe for Disaster: The Formula That Killed Wall Street

By Felix Salmon 02.23.09



$$\Pr[T_A < 1, T_B < 1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

Here's what killed your 401(k) *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick—and fatally flawed—way to assess risk. A shorter version appears on this month's cover of Wired.*

Here is what killed your 401(k)...

Li's Gaussian copula function ...

Nassim Nicholas Taleb, hedge fund manager and author of *The Black Swan*, is particularly harsh when it comes to the copula. "People got very excited about the Gaussian copula because of its mathematical elegance, but the thing never worked," he says. "Co-association between securities is not measurable using correlation," because past history can never prepare you for that one day when everything goes south. "Anything that relies on correlation is charlatanism."

Felix Salmon, Wired, February 2009

Source: <https://www.wired.com/2009/02/wp-quant/>

Financial econometrics has always been risky



Source <https://www.fynsa.com/en/newsletter/>

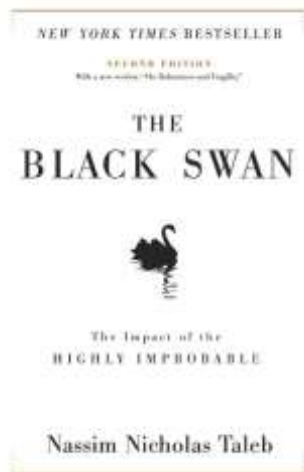
Rober Merton, Myron Scholes and Fisher Black

Scholes and Merton were awarded the 1973 (so-called) Nobel Prize in Economic Sciences (Black had passed away)

Scholes and Merton created the Long-Term Capital Management fund in 1994

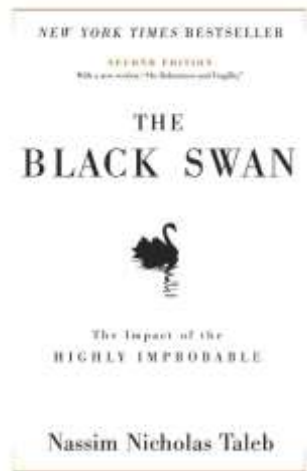
In 1998 the fund had to be bailed out by the Feds

The story and more here

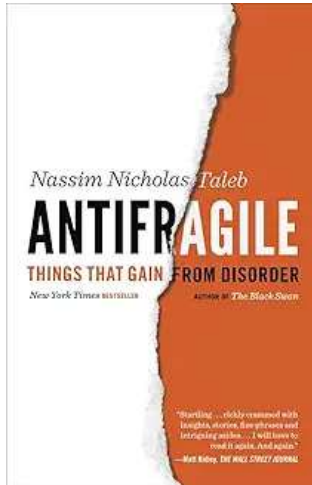


What killed LTCP

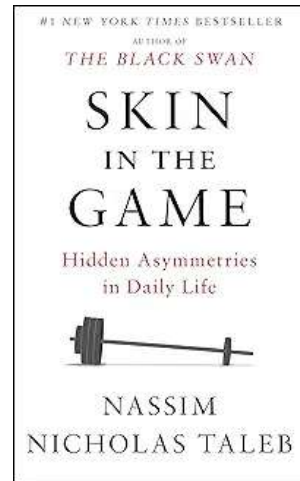
- Extreme market volatility,
- Correlated risk across seemingly unrelated assets,
- Inadequate attention to liquidity risks,
- Assumptions of normal market behaviour, and
- Excessive leverage.



2010



2014



2018

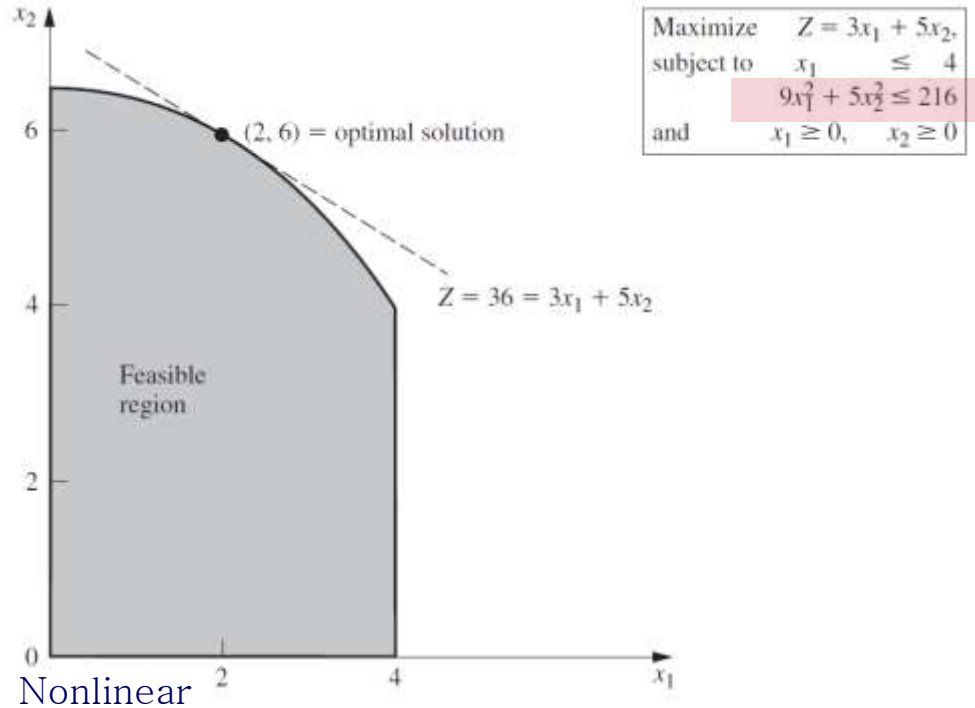
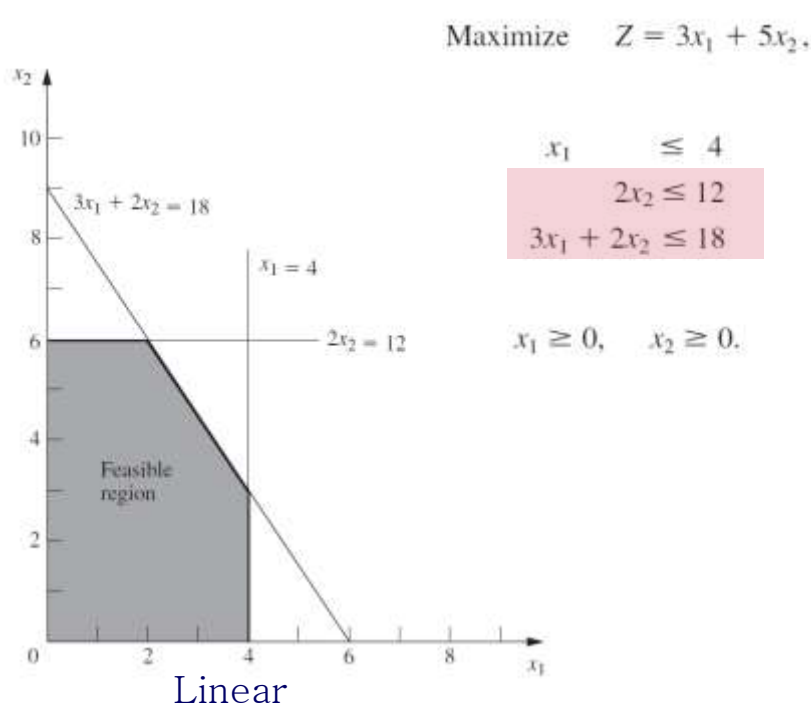
Since we mentioned him ...

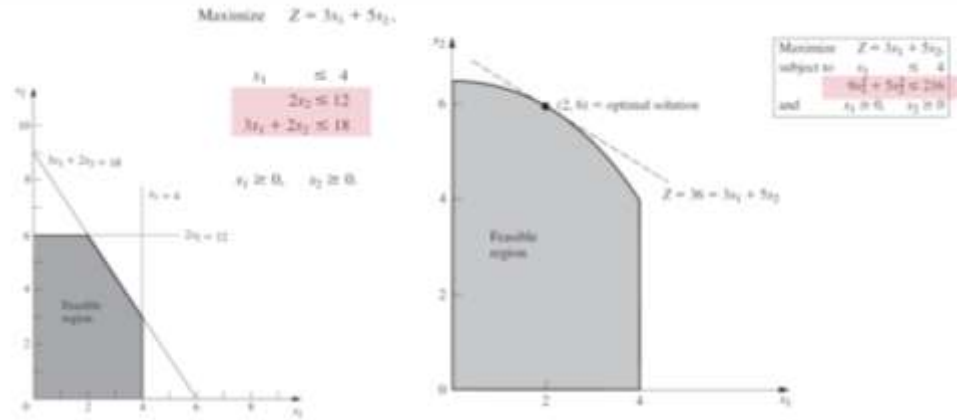
Nassim Nicholas Taleb



Source: Wikipedia Commons

Back to Hiller: an old example made nonlinear





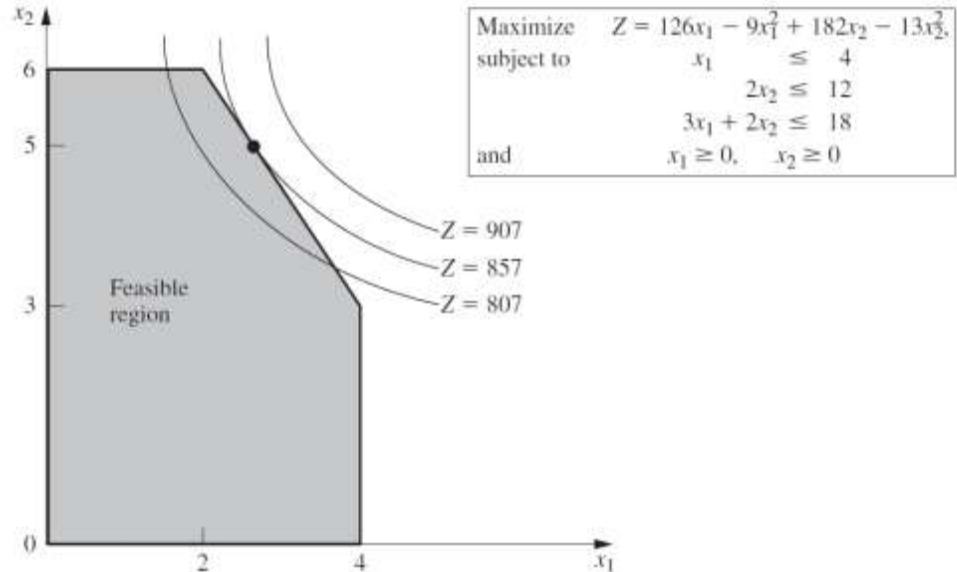
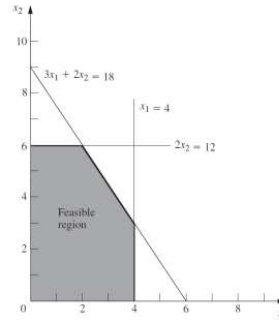
The solution still happens to be on the boundary of the feasible region, but there are no longer the corner points feasible (CPF) to help us

If we now maintain the old constraints of the linear problem i.e.

$$\begin{aligned}x_1 &\leq 4 \\2x_2 &\leq 12 \\3x_1 + 2x_2 &\leq 18 \\x_1 &\geq 0 \\x_2 &\geq 0\end{aligned}$$

But change the objective function to a nonlinear form

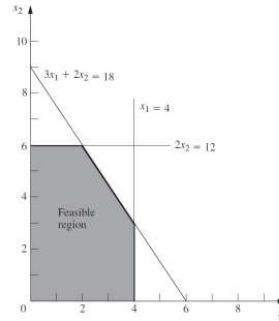
$$Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2$$



If we now maintain the old constraints of the linear problem i.e.

$$\begin{aligned}x_1 &\leq 4 \\2x_2 &\leq 12 \\3x_1 + 2x_2 &\leq 18 \\x_1 &\geq 0 \\x_2 &\geq 0\end{aligned}$$

...or to another nonlinear form

$$Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2$$


The point $x_1, x_2 = 3, 3$ is a maximum.

Differentiate with respect to x_1, x_2

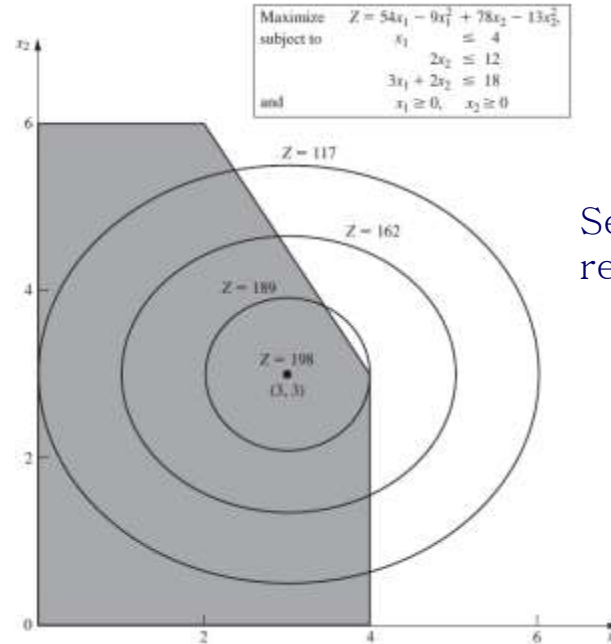
$$Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2$$

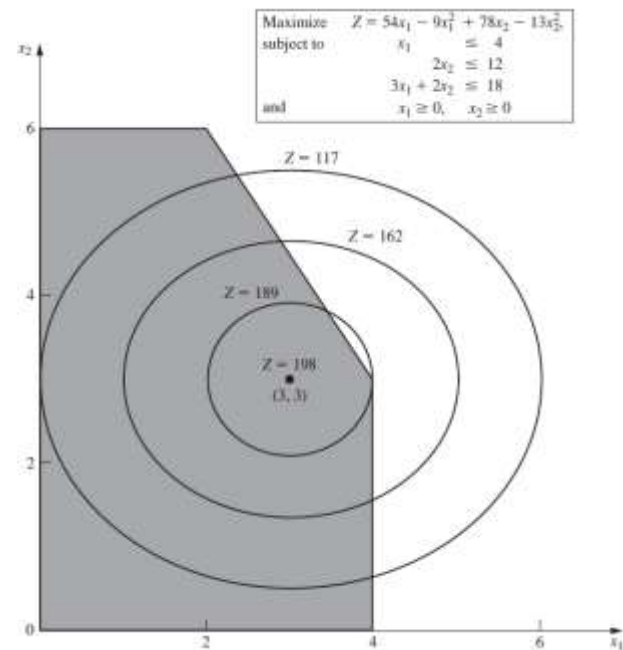
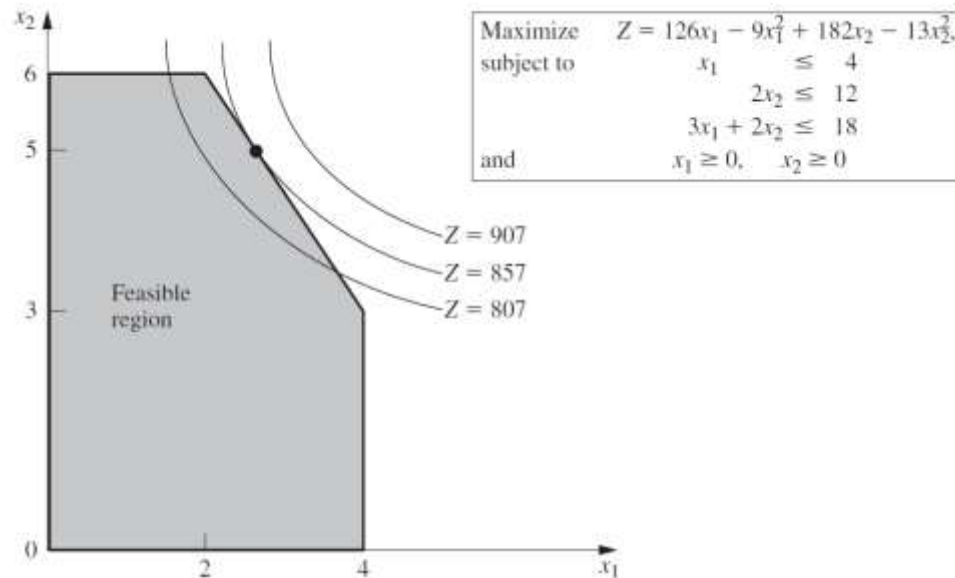
$$\begin{aligned}\frac{\partial Z}{\partial x_1} &= 54 - 18x_1 \\ \frac{\partial Z}{\partial x_2} &= 78 - 26x_2\end{aligned}$$

Set both derivatives to zero and resolve

$$\frac{\partial Z}{\partial x_1} = 0 \text{ gives } x_1 = 3$$

$$\frac{\partial Z}{\partial x_2} = 0 \text{ gives } x_2 = 3$$



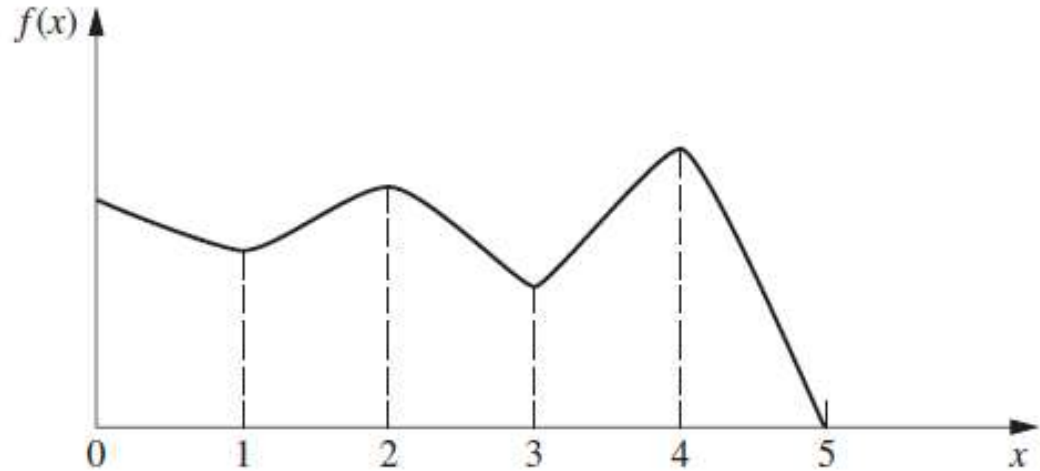


The solution can be anywhere in the feasible region, no longer just on its frontier

Other complications of the nonlinear problem: there can be more maxima

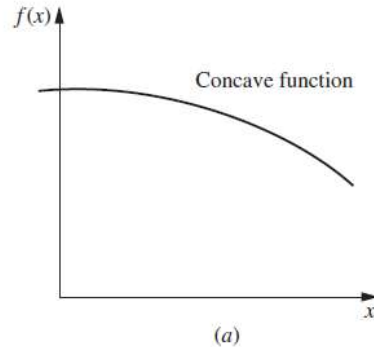
■ **FIGURE 13.8**

A function with several local maxima ($x = 0, 2, 4$), but only $x = 4$ is a global maximum.



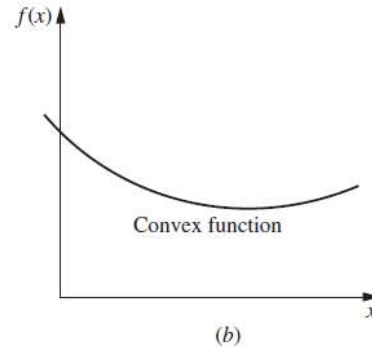
In order to have just one maximum the objective function must be **concave** over the entire dominion of the search. Similarly for a **convex** function there is guarantee of just one minimum

■ **FIGURE 13.9**
Examples of (a) a concave function and (b) a convex function.



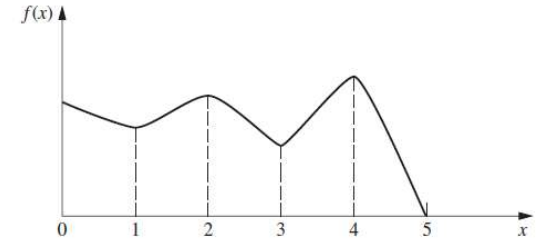
$$\frac{\partial^2 f(x)}{\partial x^2} \leq 0$$

concave

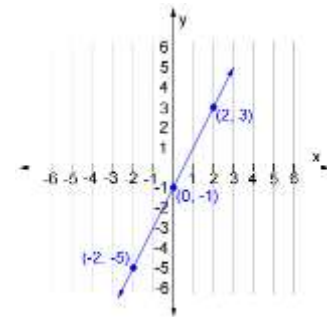


$$\frac{\partial^2 f(x)}{\partial x^2} \geq 0$$

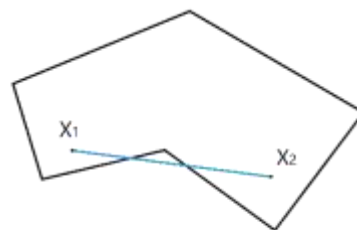
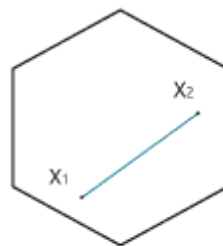
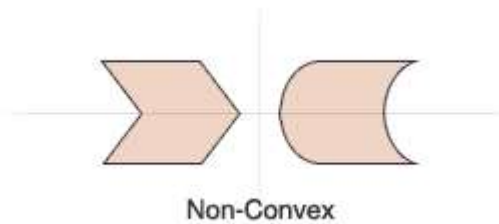
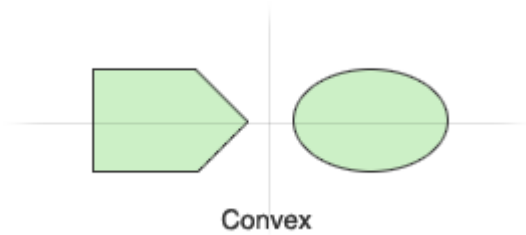
convex



Concave or convex?



Concave or convex?

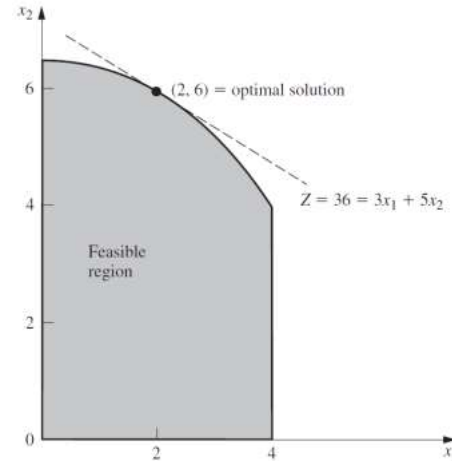
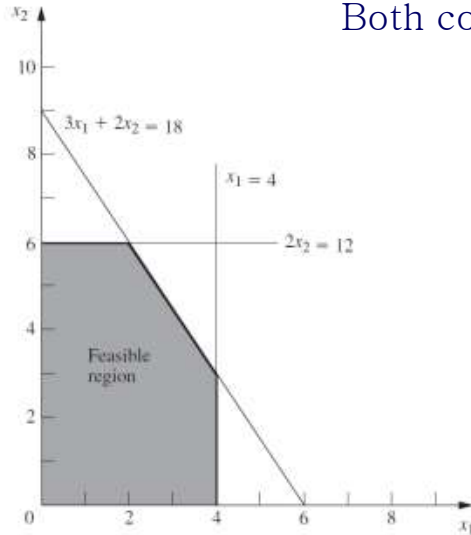


In the case of constraints the existence of a global maximum is conditioned by the shape of the feasible region. If the feasible region is a **convex set**, then the global maximum is ensured.

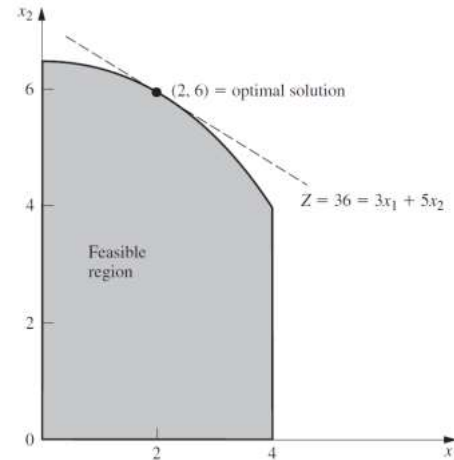
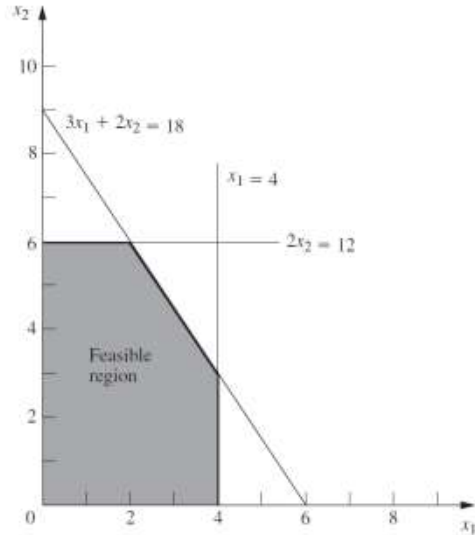
“A convex set is simply a set of points such that, for each pair of points in the collection, the entire line segment joining these two points is also in the collection”

Both convex?

Note: in linear programming the feasible region is always a convex set



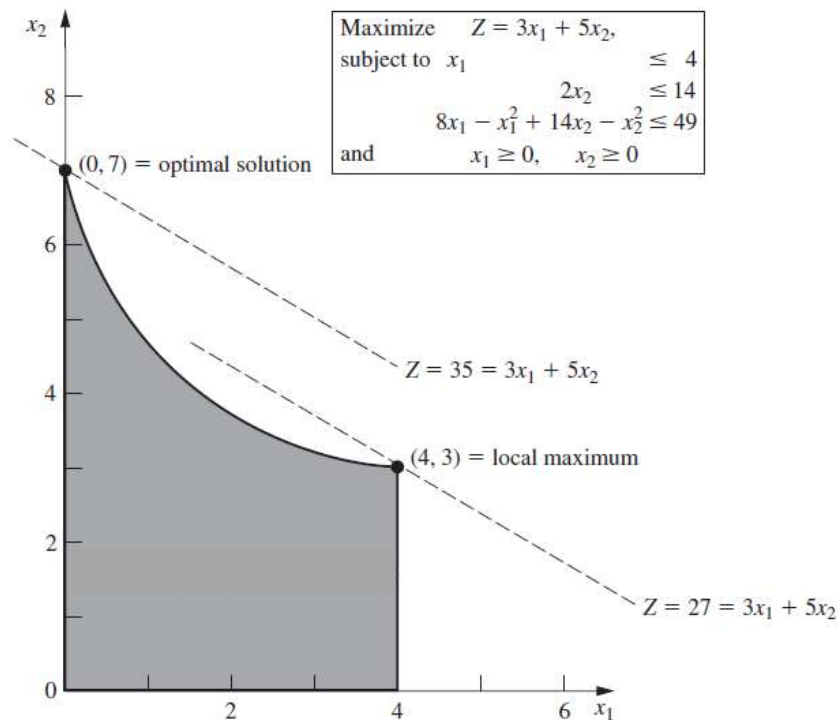
= you can reach each point in the set from each other point in the set by **walking in a straight line**



Source:
<https://mwpetersonlaw.com/>

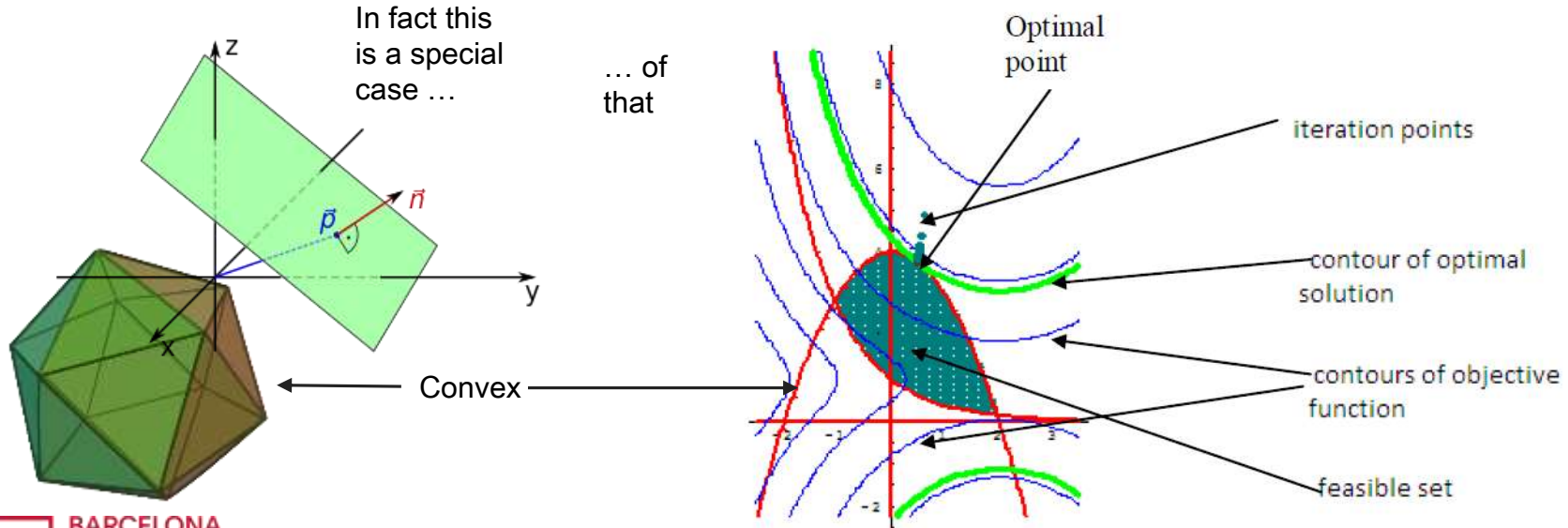
The feasible region is a convex set in both these cases

Is this feasible region convex?



A nonlinear programming problem with constraints $g_i(\mathbf{x}) \leq b_i$ $i = 1, 2, \dots, m$ has a **global maximum** when the objective $f(\mathbf{x})$ is a concave function and all the $g_i(\mathbf{x})$ are convex

➔ This is known as convex programming problem



The minimum/maximum can be anywhere;

Exercise:

- unconstrained problem, write a function $f(x)$ whose maximum is in $(x = 0)$

For example $f(x) = -x^2$

- unconstrained problem, write a function $f(x_1, x_2)$ whose minimum is in $(3, 3)$

For example $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$

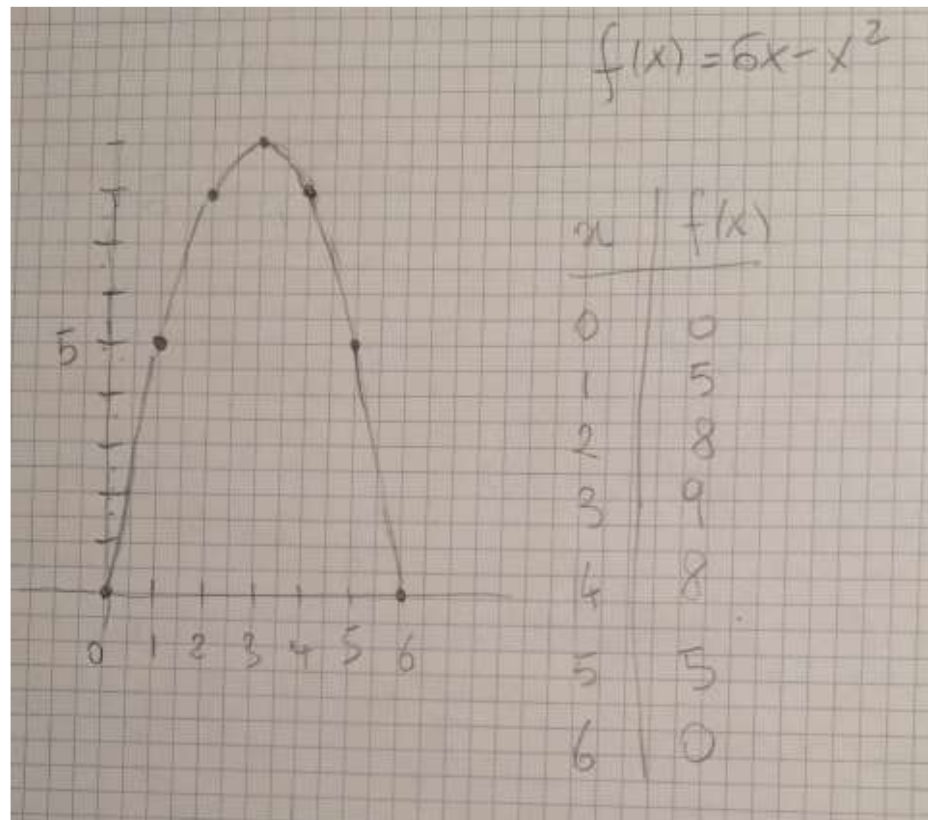
- Draw $f(x) = 6x - x^2$ between $x = 0$ and $x = 6$ and find the maximum



Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

The maximum of $f(x) = 6x - x^2$ is at $x = 3$ where:

$$\frac{\partial f(x)}{\partial x} = 6 - 2x = 0$$





Please write a program in Python to plot $6x - x^2$



To plot the function `6x - x**2` in Python, you can use the `matplotlib` library. If you haven't already installed `matplotlib`, you can do so using `pip`:



bash

Copy code

```
pip install matplotlib
```

Once you have `matplotlib` installed, you can create a simple Python script to generate the plot. Here's an example program:

python

Copy code

```
import numpy as np
import matplotlib.pyplot as plt
```

Ask ChatGTP

Go to

<https://openai.com/>

Login or create account

Ask ChatGPT to write a program in Python to plot $6x - x^2$

AN

Please write a program in Python to plot $6x - x^{**2}$

Install Anaconda from
<https://www.anaconda.com/download>



To plot the function `6x - x**2` in Python, you can use the `matplotlib` library. If you haven't already installed `matplotlib`, you can do so using `pip`:

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python

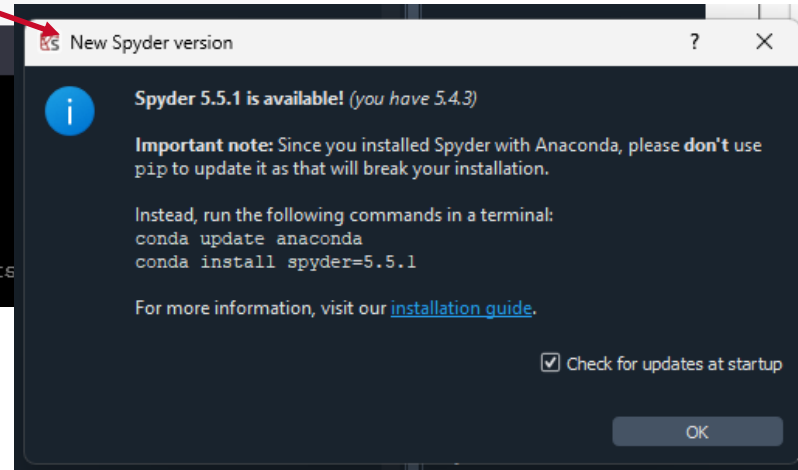
```
import numpy as np
import matplotlib.pyplot as plt

# Define the range of x values
x = np.linspace(0, 6, 100) # Generate 100 evenly spaced points
```




Launch Anaconda and
Spyder

ChatGPT and Spyder-Python
disagree here – tell me what
you find



python

 Copy code

```
import matplotlib.pyplot as plt
import numpy as np

# Define the x values
x = np.linspace(0, 6, 100) # Generate 100 evenly spaced points between 0 and 6

# Calculate the corresponding y values using the function 6x - x^2
y = 6 * x - x**2

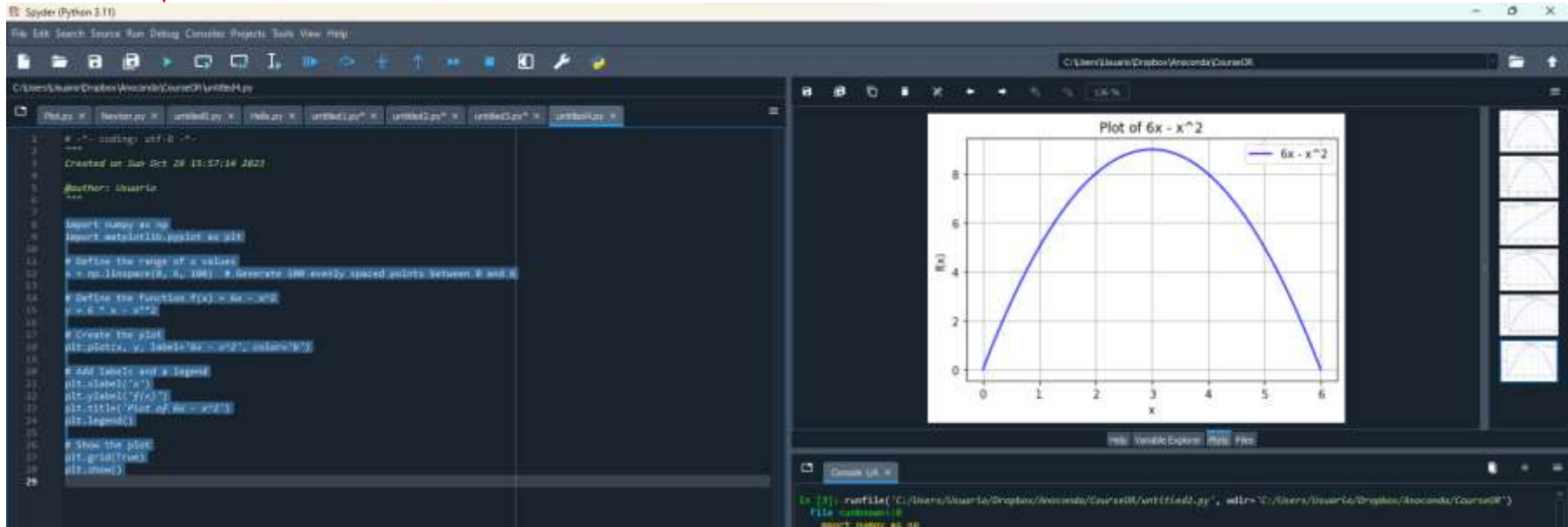
# Create a plot
plt.plot(x, y, label='6x - x^2')

# Add labels and a legend
plt.xlabel('x')
plt.ylabel('y')
plt.title('Plot of 6x - x^2')
plt.legend()

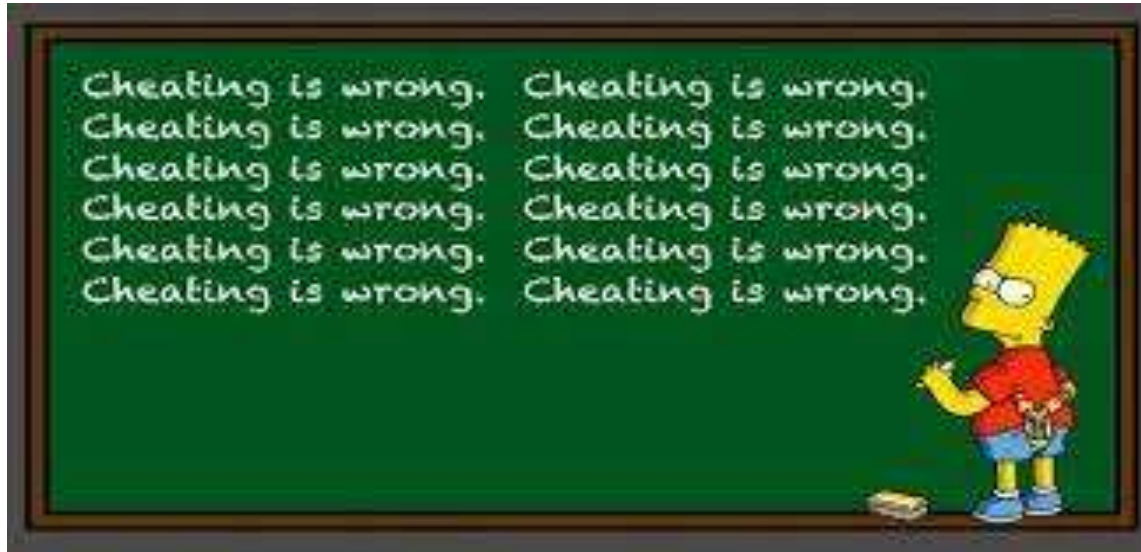
# Show the plot
plt.grid(True)
plt.show()
```

Execute

Execute



Digression: is using ChatGPT cheating?



Source: The Simpsons, Twentieth Century Fox Film Corporation

Next slides ‘scary’? Don’t panic



Not for the final exam

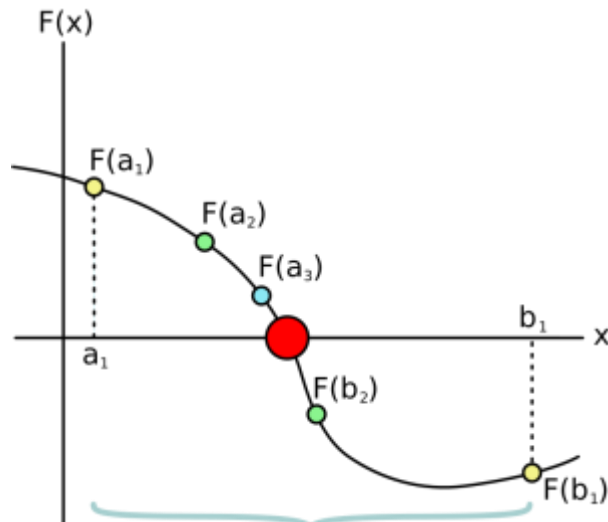


Bisection method (Bolzano search plan) – for concave functions.

If a continuous function has values of opposite sign inside an interval, then it has a root in that interval (Bolzano's theorem)

Not rocket science!

I can get closer and closer to the zero by choosing iteratively the midpoints ...



Bernardus Placidus
Johann Nepomuk
Bolzano, 1781–1848
Source: Wikipedia
Commons

Bisection method (Bolzano search plan) – for concave functions.

If a continuous function has values of opposite sign inside an interval, then it has a root in that interval (Bolzano's theorem)

Applying this theorem to the derivative $\frac{\partial f(x)}{\partial x}$, knowing that for concave functions the maximum corresponds to the point where $\frac{\partial f(x)}{\partial x} = 0$ one can first identify an interval where $\frac{\partial f(x)}{\partial x}$ changes sign, then reduce the dimension interval iteratively to get the solution



Bernardus Placidus
Johann Nepomuk
Bolzano, 1781–1848
Source: Wikipedia
Commons

Notation

x^* solution being sought

ε tolerance in the search of x^*

x' current trial solution

x_l current lower bound

x_u current upper bound

We want the maximum of

$$f(x) = 6x - x^2 - \frac{x^3}{3}$$

Setting the tolerance ε at
0.1 (10%)

Procedure: Find extreme x_u and x_l so that $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x_l} < 0$, while $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x_u} > 0$
and initialize $x' = \frac{x_l + x_u}{2}$

1) Evaluate $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x'}$ at $x = x'$

2) if $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x'} < 0$ redefine $x_l = x'$

3) if $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x'} > 0$ redefine $x_u = x'$

4) Update $x' = \frac{x_l + x_u}{2}$

Iterate 1-4 till $x_u - x_l \leq 2\varepsilon$ so that new x' must be within ε of x^*

Use the procedure to find
a local maximum of

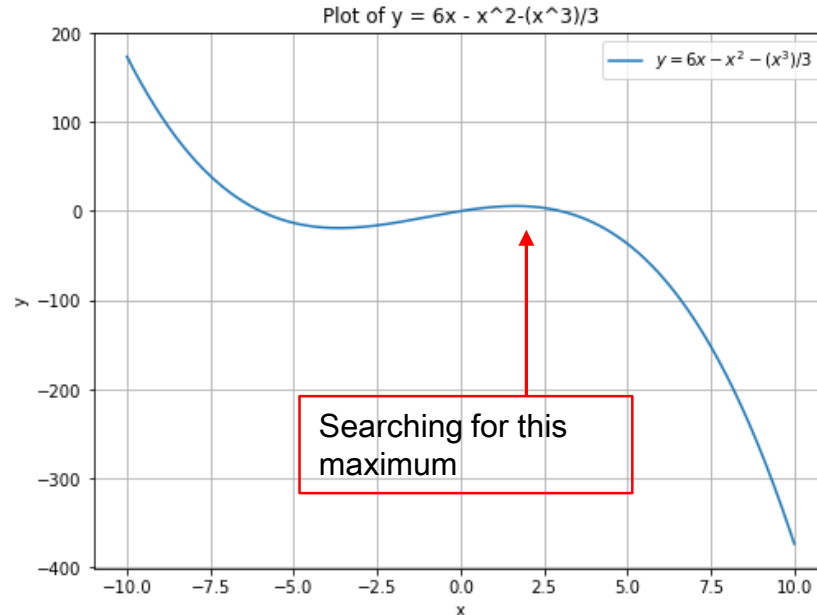
$$f(x) = 6x - x^2 - \frac{x^3}{3}$$

Between 1. and 2. Set the
tolerance ε at 0.1 (10%)

Help: $\frac{\partial f(x)}{\partial x} = 6 - 2x - x^2$



Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous



Procedure: Find extreme x_u and x_l so that $\frac{\partial f(x)}{\partial x}\bigg|_{x=x_l} < 0$, while $\frac{\partial f(x)}{\partial x}\bigg|_{x=x_u} > 0$ and initialize $x' = \frac{x_l + x_u}{2}$

- 1) Evaluate $\frac{\partial f(x)}{\partial x}$ at $x = x'$
- 2) if $\frac{\partial f(x)}{\partial x}\bigg|_{x=x'} < 0$ redefine $x_l = x'$
- 3) if $\frac{\partial f(x)}{\partial x}\bigg|_{x=x'} > 0$ redefine $x_u = x'$
- 4) Update $x' = \frac{x_l + x_u}{2}$

Iterate 1-4 ...

Use the procedure to find a local maximum of

$$f(x) = 6x - x^2 - \frac{x^3}{3}$$

Between 1. and 2.

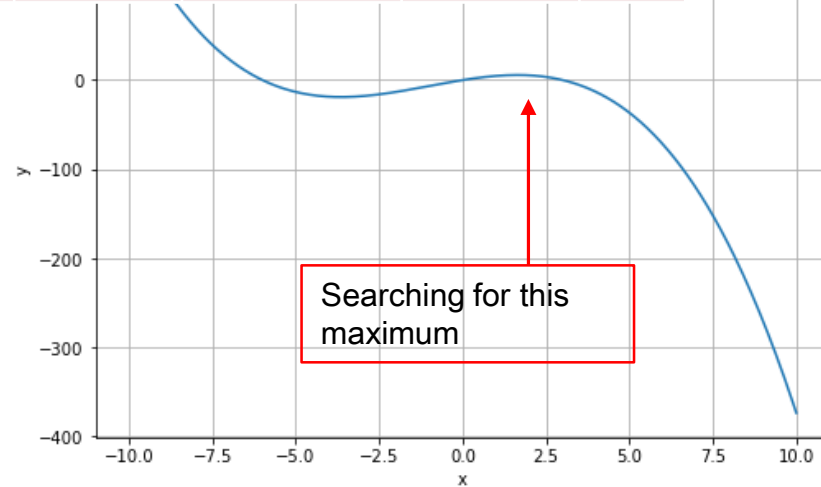
Help: $\frac{\partial f(x)}{\partial x} = 6 - 2x - x^2$

x	$\frac{\partial f(x)}{\partial x}$
1	3
2	-2

x_l	x_u	$\frac{\partial f(x)}{\partial x}\bigg _{x=x_l}$	$\frac{\partial f(x)}{\partial x}\bigg _{x=x_u}$	x'
1	2	3	-2	1.5



$$6x - x^2 - \frac{x^3}{3}$$



Procedure: Find extreme x_u and x_l so that $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x_l} < 0$, while $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x_u} > 0$ and initialize $x' = \frac{x_l + x_u}{2}$

- 1) Evaluate $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x'}$ at $x = x'$
- 2) if $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x'} < 0$ redefine $x_l = x'$
- 3) if $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x'} > 0$ redefine $x_u = x'$
- 4) Update $x' = \frac{x_l + x_u}{2}$

Iterate 1-4 ...

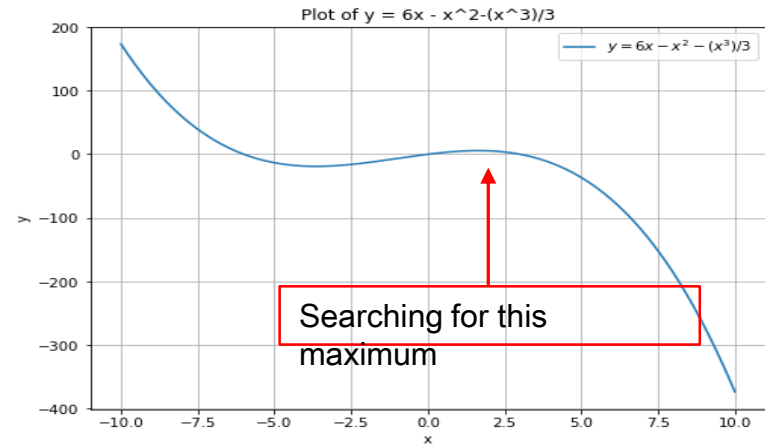
Use the procedure to find a local maximum of

$$f(x) = 6x - x^2 - \frac{x^3}{3}$$

Between 1. and 2.

Help: $\frac{\partial f(x)}{\partial x} = 6 - 2x - x^2$

x_l	x_u	$\left. \frac{\partial f(x)}{\partial x} \right _{x=x_l}$	$\left. \frac{\partial f(x)}{\partial x} \right _{x=x_u}$	x'	$\left. \frac{\partial f(x)}{\partial x} \right _{x=x'}$
1	2	3	-2	1.5	0.75
1.5	2	0.75	-2	$\frac{1.5 + 2}{2}$...



By hand, I get $x=1.72$, $\frac{\partial f(x)}{\partial x} \sim 0.4$
after 3 iterations and some
number crunching; search
interval $x \in (1,2)$

The procedure is laborious
(slow convergence)

Procedure

Find extreme x_u and x_l so that $\frac{\partial f(x)}{\partial x} \Big|_{x=x_l} < 0$, while
 $\frac{\partial f(x)}{\partial x} \Big|_{x=x_u} > 0$ and initialize $x' = \frac{x_l + x_u}{2}$

- 1) Evaluate $\frac{\partial f(x)}{\partial x}$ at $x = x'$
- 2) if $\frac{\partial f(x)}{\partial x} \Big|_{x=x'} < 0$ redefine $x_l = x'$
- 3) if $\frac{\partial f(x)}{\partial x} \Big|_{x=x'} > 0$ redefine $x_u = x'$
- 4) Update $x' = \frac{x_l + x_u}{2}$

Iterate 1-4 till $x_u - x_l \leq 2\varepsilon$ so that new x' must be
within ε of x^*

Newton's method

Newton is credited to have discovered calculus (in parallel with Leibniz) and his method is that of a quadratic approximation based on a truncated Taylor series

Isac Newton (1643–1727)



Source: Wikipedia Commons

$$f(x_{i+1}) = f(x_i) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_i} (x_{i+1} - x_i) + \frac{1}{2} \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_i} (x_{i+1} - x_i)^2 + \frac{1}{6} \left. \frac{\partial^3 f(x)}{\partial x^3} \right|_{x=x_i} (x_{i+1} - x_i)^3 \cdots + \frac{1}{n!} \left. \frac{\partial^n f(x)}{\partial x^n} \right|_{x=x_i} (x_{i+1} - x_i)^n$$

We stop (truncate) now at the second order term

Looking at this as a function of only x_{i+1} , with x_i and its derivative as **fixed**

$$f(x_{i+1}) = f(x_i) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_i} (x_{i+1} - x_i) + \frac{1}{2} \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_i} (x_{i+1} - x_i)^2 + \dots$$

$$f(u) = f(x_i) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_i} (u - x_i) + \frac{1}{2} \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_i} (u - x_i)^2 + \dots$$

← Just renamed x_{i+1} as u

Putting $\frac{\partial f(u)}{\partial u} = 0$ we get: $\left. \frac{\partial f(x)}{\partial x} \right|_{x=x_i} + \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_i} (x_{i+1} - x_i)$

Reordering we get this beauty

$$x_{i+1} = x_i - \frac{\left. \frac{\partial f(x)}{\partial x} \right|_{x=x_i}}{\left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_i}}$$



Using the beauty $x_{i+1} = x_i - \frac{\frac{\partial f(x)}{\partial x}|_{x=x_i}}{\frac{\partial^2 f(x)}{\partial x^2}|_{x=x_i}}$

The procedure is simple: find a trial value x_i by inspection

1) Evaluate $\frac{\partial f(x)}{\partial x}|_{x=x_i}$

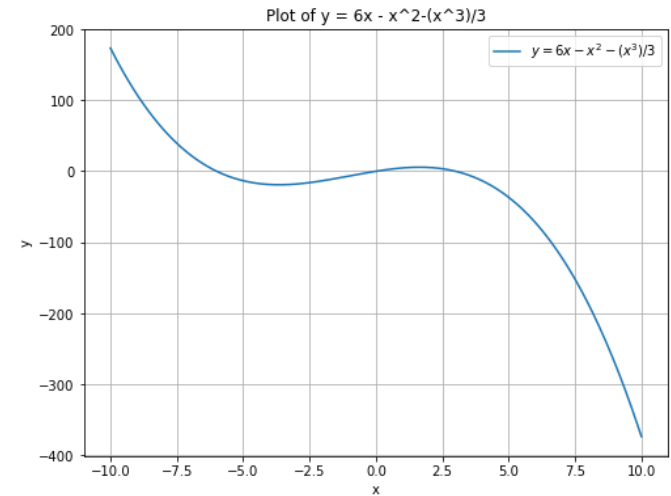
2) Evaluate $\frac{\partial^2 f(x)}{\partial x^2}|_{x=x_i}$

3) Compute x_{i+1} as from the formula above

Iterate 1-3 till $|x_{i+1} - x_i| \leq \varepsilon$

Exercise: Compute at least up to the first iteration, starting from $x_i = 1$ (at home, facultative)

$$f(x) = 6x - x^2 - \frac{x^3}{3}; \quad \frac{\partial f(x)}{\partial x} = 6 - 2x - x^2; \quad \frac{\partial^2 f(x)}{\partial x^2} = -2 - 2x$$



Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

Using this beauty $x_{i+1} = x_i - \frac{\frac{\partial f(x)}{\partial x} \Big|_{x=x_i}}{\frac{\partial^2 f(x)}{\partial x^2} \Big|_{x=x_i}}$

The procedure is simple: find a trial value x_i by inspection

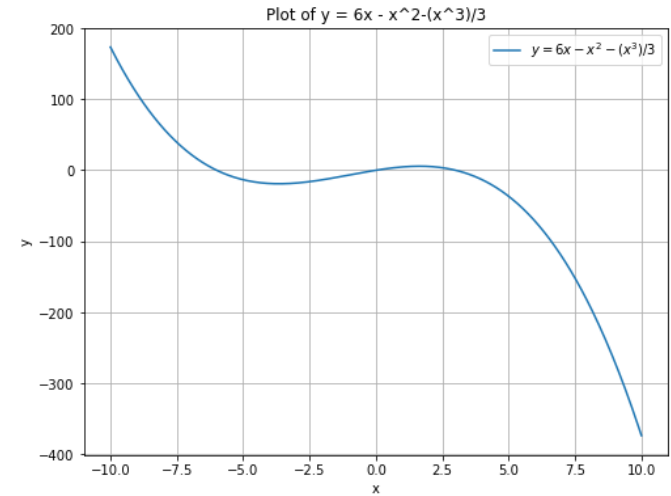
1) Evaluate $\frac{\partial f(x)}{\partial x} \Big|_{x=x_i}$

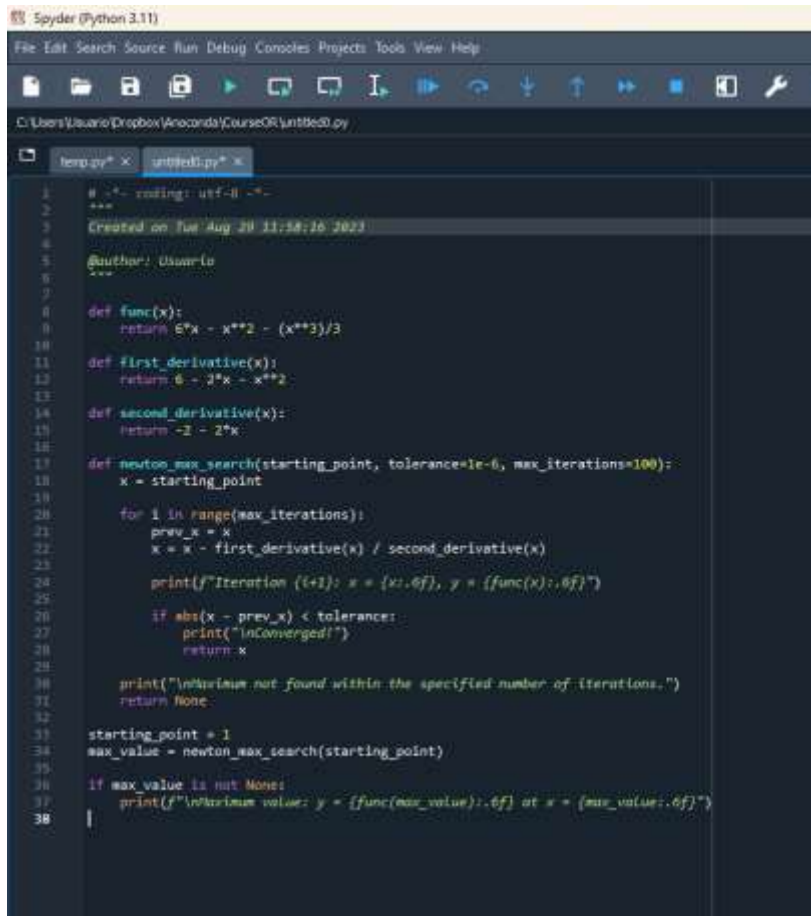
2) Evaluate $\frac{\partial^2 f(x)}{\partial x^2} \Big|_{x=x_i}$

3) Compute x_{i+1} as from the formula above

Iterate 1-3 till $|x_{i+1} - x_i| \leq \varepsilon$, starting from $x_i = 1$

Phyton code can be used again





```
1 # -*- coding: utf-8 -*-
2 """
3 Created on Tue Aug 20 11:58:26 2023
4
5 @author: Usman-Lo
6 """
7
8 def func(x):
9     return 6*x - x**2 - (x**3)/3
10
11 def first_derivative(x):
12     return 6 - 2*x - x**2
13
14 def second_derivative(x):
15     return -2 - 2*x
16
17 def newton_max_search(starting_point, tolerance=1e-6, max_iterations=100):
18     x = starting_point
19
20     for i in range(max_iterations):
21         prev_x = x
22         x = x - first_derivative(x) / second_derivative(x)
23
24         print(f"Iteration {i+1}: x = {x:.6f}, y = {func(x):.6f}")
25
26         if abs(x - prev_x) < tolerance:
27             print("Converged!")
28             return x
29
30     print("Maximum not found within the specified number of iterations.")
31     return None
32
33 starting_point = 1
34 max_value = newton_max_search(starting_point)
35
36 if max_value is not None:
37     print(f"Maximum value: y = {func(max_value):.6f} at x = {max_value:.6f}")
38 |
```

Question: Please find the maximum of the function $y=6x-x^2-(x^3)/3$, using the method of Newton pure, using the first and the second derivative, with starting point $x=1$, and printing out all steps of the search.



To find the maximum of the function $y = 6x - x^2 - \frac{x^3}{3}$ using the Newton-Raphson method with the first and second derivatives, you can follow these steps:

1. Define the function $y = 6x - x^2 - \frac{x^3}{3}$.
2. Calculate the first and second derivatives of the function.
3. Implement the Newton-Raphson method using the formula: $x_{\text{new}} = x_{\text{old}} - \frac{f'(x_{\text{old}})}{f''(x_{\text{old}})}$.
4. Iterate until the change in x is smaller than a certain tolerance or a maximum number of iterations is reached.
5. Print out each iteration's x value and the corresponding function value.

Here's how you can implement this in Python:

Iteration 1: $x = 1.750000$, $y = 5.651042$
Iteration 2: $x = 1.647727$, $y = 5.680162$
Iteration 3: $x = 1.645752$, $y = 5.680173$
Iteration 4: $x = 1.645751$, $y = 5.680173$

A different model, where we explore rather than optimize

The Bateman equations, initially written for a radioactive decay chain, can be used to describe any process where transfer occur among compartments without backward term

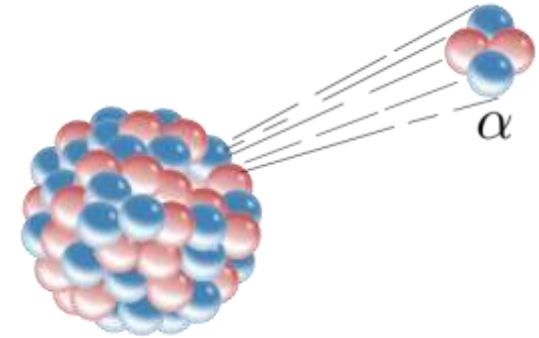
$\partial N_1 / \partial t = -\lambda_1 N_1$ N_1 concentration of species 1, disappear proportionally to itself ...

$\partial N_2 / \partial t = \lambda_1 N_1 - \lambda_2 N_2$...to become N_2 that in turn disappear proportionally to itself...

... ...and so on...

$\partial N_n / \partial t = \lambda_{n-1} N_{n-1} - \lambda_n N_n$...till and end-product is reached

We choose this as it is nonlinear and the differential equation has a neat analytic solution



Source: Wikipedia common

$$\partial N_1 / \partial t = -\lambda_1 N_1$$

$$\partial N_2 / \partial t = \lambda_1 N_1 - \lambda_2 N_2$$

...

$$\partial N_n / \partial t = \lambda_{n-1} N_{n-1} - \lambda_n N_n$$

For $N_1(0) \neq 0$, $N_i(0) = 0 \forall i \neq 1$ the solution is

$$N_k(t) = \frac{N_1(0)}{\lambda_k} \sum_{i=1}^k \lambda_i \alpha_i e^{-\lambda_i t}$$

With

$$\alpha_i = \prod_{j=1, j \neq i}^k \frac{\lambda_j}{\lambda_j - \lambda_i}$$



Bateman equations in Python

Interested in playing
with this function in
Python? Script in
eCampus

A last model, where we revisit our simple linear form (from Lesson 3): $y = \sum_{i=1}^k \Omega_i Z_i$

Where y (a scalar) is the output of interest, the Ω_i 's **were** fixed coefficients and Z_i 's are uncertain input factors following a Normal distribution

$$Z_i \sim N(\bar{z}_i, \sigma_{Z_i})$$

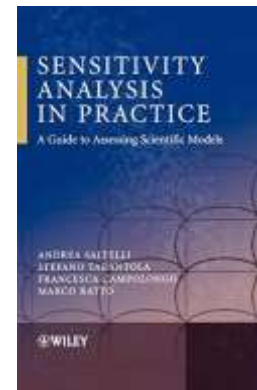
Where $\bar{z}_i = 0, i = 1, 2, \dots, k$ are the means of the factors Z_i 's and σ_i their standard deviations

We now allow the Ω_i to **be uncertain as well**

$\Omega_i \sim N(\bar{\Omega}_i, \sigma_{\Omega_i})$ where $\bar{\Omega}_i = 0, i = 1, 2, \dots, k$ are the means of the factors Ω_i 's and σ_{Ω_i} their standard deviations

Interested in playing
with this function in
Python? Script in
eCampus

In this book we took \bar{z}_i to be zero, and called $y = \sum_{i=1}^k \Omega_i Z_i$ a balanced portfolio, where the Z_i are the assets and the Ω_i the amount held of each security



There are as well stochastic search method (called meta-heuristics in the Hillier's book) where the search is done iteratively with trial points and rules to point the search in the right direction, without being greedy

- Tabu Search (don't go there if you have been there already)
- Simulated Annealing (you can walk in the wrong direction but with lower probability)
- Genetic Algorithms (let the fitter reproduce themselves)



We study this

Digression: Genetic algorithms (let the **fitter** reproduce themselves)

Note: Darwin used ‘fittest’, borrowing the term from Spencer, but ‘fitter’ is more apt to his theory as well as to what genetic algorithms do.

“Darwin did not consider the process of evolution as the survival of the fittest; he regarded it as survival of the fitter, because the “struggle for existence” is relative and thus not absolute. Instead, the winners with respect to species within ecosystems could become losers with a change of circumstances”
(<https://www.britannica.com>)



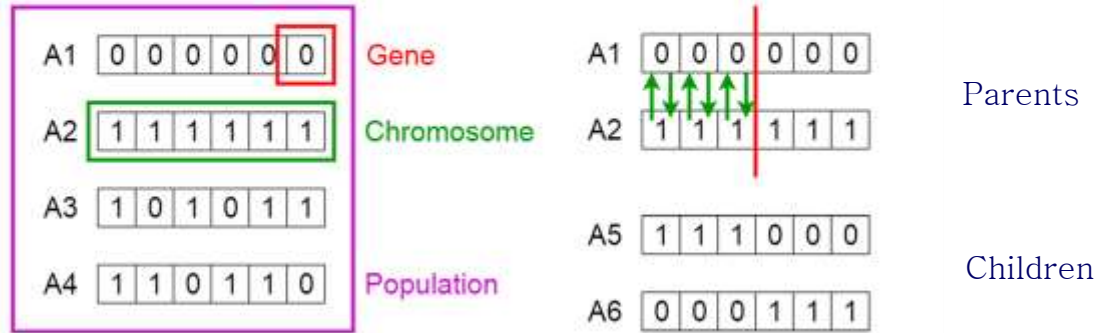
Herbert Spencer
(1820–1903)



Charles Darwin
(1809–1882)

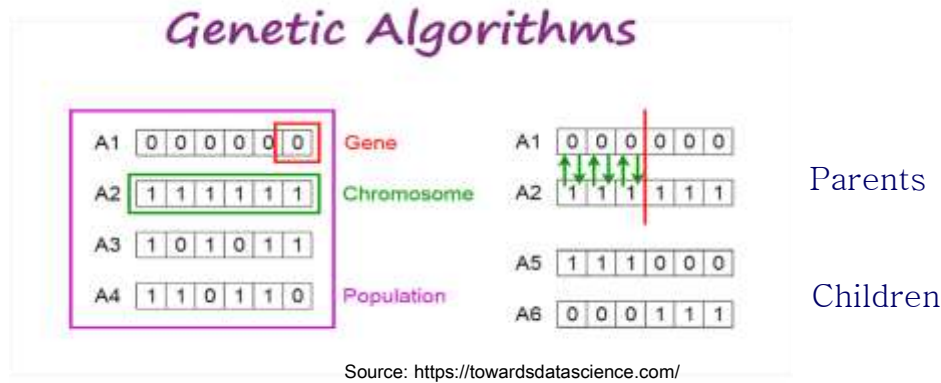
Unlike Tabu search and Simulated Annealing, genetic algorithms do not work with a wandering point, but with an evolving population – a collection of candidate points is generated right at the start, then these generate offspring

Genetic Algorithms

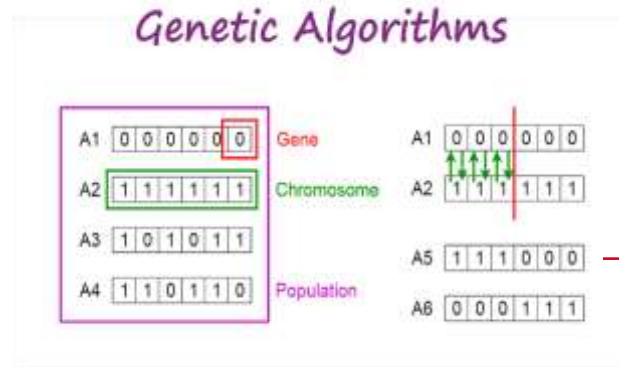


Source: <https://towardsdatascience.com/>

At each generation the parents with the higher fitness have higher probability of reproducing, with each parent passing part of his genes to the offspring → fitness in terms of objective function Z



Additional random mutations can occur ('errors' in the transcription of DNA or epigenetic factors in the genetic metaphor)



Parents

Children

Before Mutation

A5 1 1 1 0 0 0

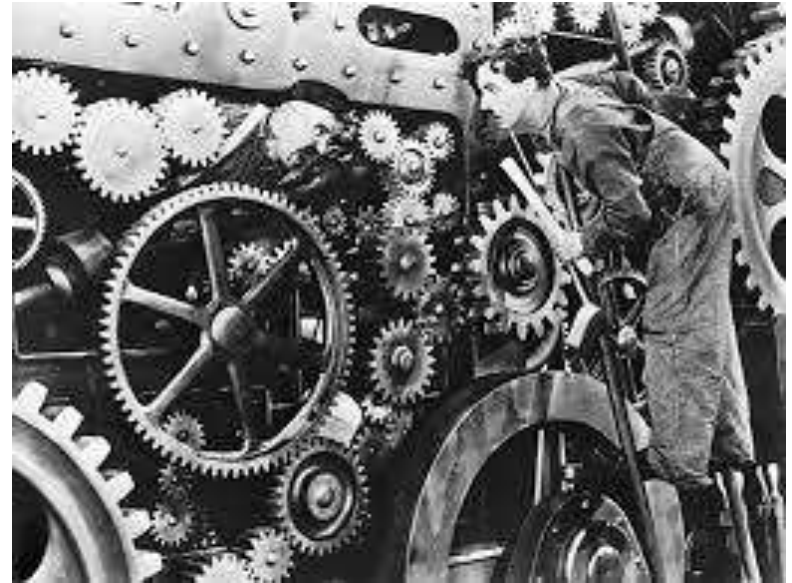
After Mutation

A5 1 1 0 1 1 0

Source: <https://towardsdatascience.com/>

Genetic algorithms have lots of movable parts! In order to use them you must decide

- Size and composition of the initial population
- How to select the parents based on Z
- How to exchange the genes of the parent to generate the children
- Mutation rate
- Stopping rule



Source: Charlie Chaplin's Modern Times

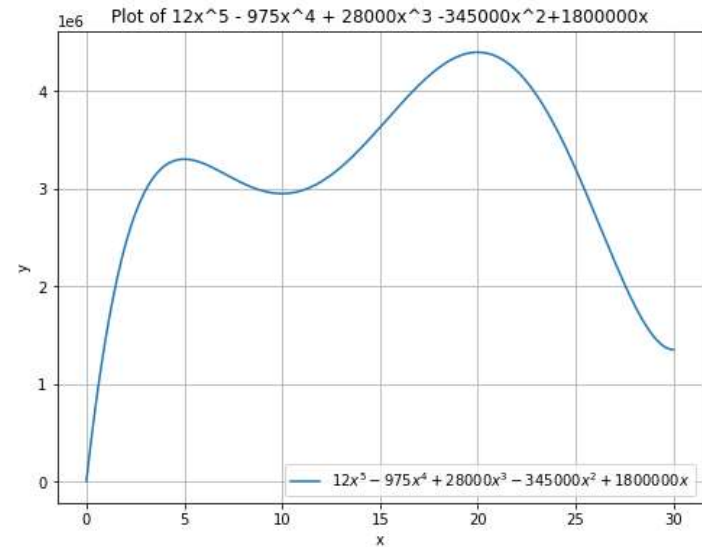
Try with GA with this function (Hillier, Chapter #14)
 $y = 12x^5 - 975x^4 + 28,000x^3 - 345,000x^2 + 1,800,000x^5$

```
Spyder (Python 3.11)
File Edit Search Source Run Debug Consoles Projects Tools View Help

C:\Users\luana\Dropbox\ANACORCA\Course03\Pol.py

Pol.py * Newbin.py * init0ed0.py *

1  # -*- coding: utf-8 -*-
2  """
3  Spyder Editor
4  This is a temporary script file.
5  """
6
7
8  import numpy as np
9  import matplotlib.pyplot as plt
10
11  def func(x):
12      return 12*x**5 - 975*(x**4) + 28000*(x**3) - 345000*(x**2) + 1800000*x
13
14  # Generate x values
15  x = np.linspace(0, 30, 400)
16
17  # Calculate corresponding y values
18  y = func(x)
19
20  # Create the plot
21  plt.figure(figsize=(10, 5))
22  plt.plot(x, y, label=r'12x^5 - 975x^4 + 28000x^3 - 345000x^2 + 1800000x')
23  plt.title('Plot of 12x^5 - 975x^4 + 28000x^3 - 345000x^2 + 1800000x')
24  plt.xlabel('x')
25  plt.ylabel('y')
26  plt.grid(True)
27  plt.legend()
28  plt.show()
29
```



Plotted with Python generated by ChatGPT

Try with GA with the monster (Hillier, Chapter #14)
 $y = 12x^5 - 975x^4 + 28,000x^3 - 345,000x^2 + 1,800,000x$

Since the x axis spans from zero to 32, we can represent the possible solutions in a nice binary notation

00=000000

01=000001= 2^0

02=000010= 2^1

03=000011= $2^1 + 2^0$

04=000100= 2^2

...

07=000111= $2^2 + 2^1 + 2^0$

08=001000= 2^3

...

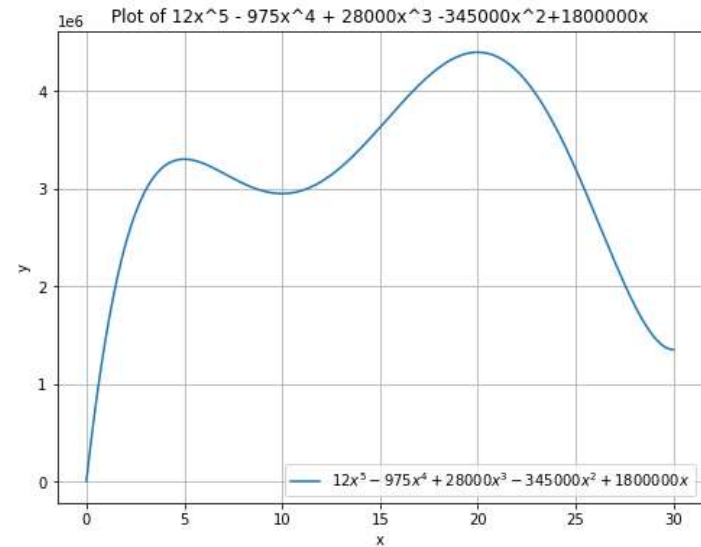
15=001111= $2^3 + 2^2 + 2^1 + 2^0$

16=010000= 2^4

31=011111= $2^4 + 2^3 + 2^2 + 2^1 + 2^0$

32=100000= 2^5

...



Refresh binary numbers

Never met binary?

$$00=000000$$

$$01=000001=2^0$$

$$02=000010=2^1$$

$$03=000011=2^1 + 2^0$$

$$04=000100=2^2$$

...

$$07=000111=2^2 + 2^1 + 2^0$$

$$08=001000=2^3$$

...

$$15=001111=2^3 + 2^2 + 2^1 + 2^0$$

$$16=010000=2^4$$

...

$$31=011111=2^4 + 2^3 + 2^2 + 2^1 + 2^0$$

$$32=100000=2^5$$

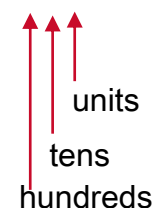
$$64 = 1000000=2^6$$

$$128=10000000=2^7$$

From Roman to Arabic numbers

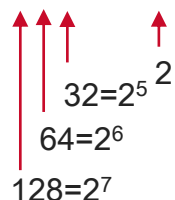
CCXXVI is conceptually **far** from

226



While binary and decimal are close

11100010 =226



Source: Wikipedia Commons

Leonardo of Pisa,
said Fibonacci
1170-1250

Never met binary?

$$\begin{aligned}0.1 &= 1/2 \\ 0.01 &= 1/4 \\ 0.001 &= 1/8\end{aligned}$$

$$0.111 = 0.875$$

$$11.001 = 3.125$$

...



Who introduced binary?

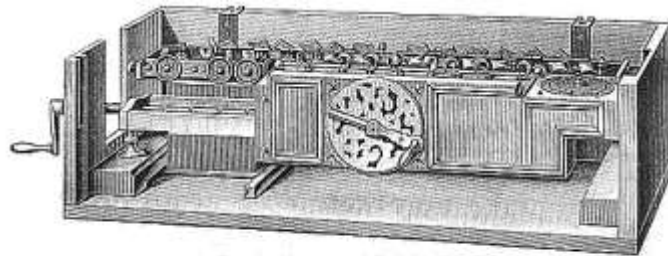
... and in Europe?



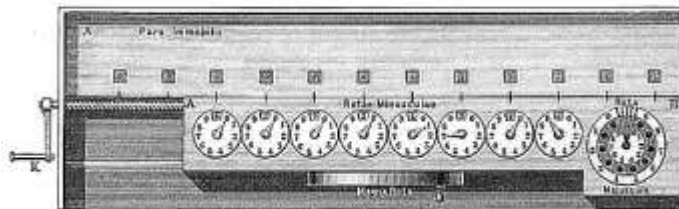
~1000 BCE



Gottfried Wilhelm Leibniz
1646–1716



2. Rechenmaschine von Leibniz (1673, Hannover).



1. Leibnizsche Rechenmaschine, geometrische Zeichnung.

From *Calculus ratiocinator* to
Characteristica Universalis

Then we have to cook some rules, e.g.

Starting population: 10 individuals (for this problem)

Choose the five fittest and the two least fit for **matching** (coupling randomly)

Switching the genes: keep repeated genes and switch the different ones

Mutation rate 1/10

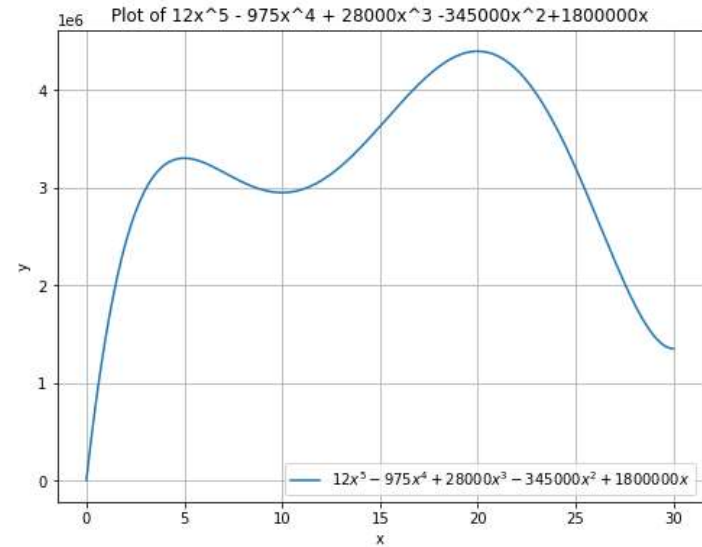
Stopping rule after five iterations without improvements



Lots of moving parts;
these are the choices
suggested in Hillier
but other choices are
possible



Source: Charlie Chaplin's Modern Times



One possible way of matching

Parents

011000

110101

Repeated genes are passed to the next generation

Children

x1xx0x

x1xx0x

How to choose the missing x's? **Russian roulette**

x1xx0x

x1xx0x

Generate a random number between 0 and 1; if between 0 and .4999 replace the first x with a zero, if between .5 and 1 replace it with a one

Where the name comes from – from revolver to spinning wheel



Source: From movie The Deer Hunter, Source: Wikipedia commons



Source: <https://www.gettyimages.es>



Source: <https://wordwall.net/>

How to choose the x's? Russian roulette

x1xx0x

x1xx0x

Generate a random number between 0 and 1; if between 0 and .4999 replace the first x with a zero, if between .5 and 1 replace it with a one

This was done and we have children

011000

010101

How to mutate 10% of this? Russian roulette: if the random number is between 0 and 0.0999 choose to mutate, otherwise no mutations and pass to the next couple;

Russian roulette also to decide which gene to change; if there are 12 (6+6) candidate genes (two parent of five genes each), **how do we do this time?**

If between 0 and $\frac{1}{12}$ mutate the first gene of the first child, in between $\frac{1}{12}$ and $\frac{2}{12}$ the second gene of the first child ... if between $\frac{11}{12}$ and 1 the last gene of the second child



Source: <https://www.gettyimages.es>



Source: <https://wordwall.net/>

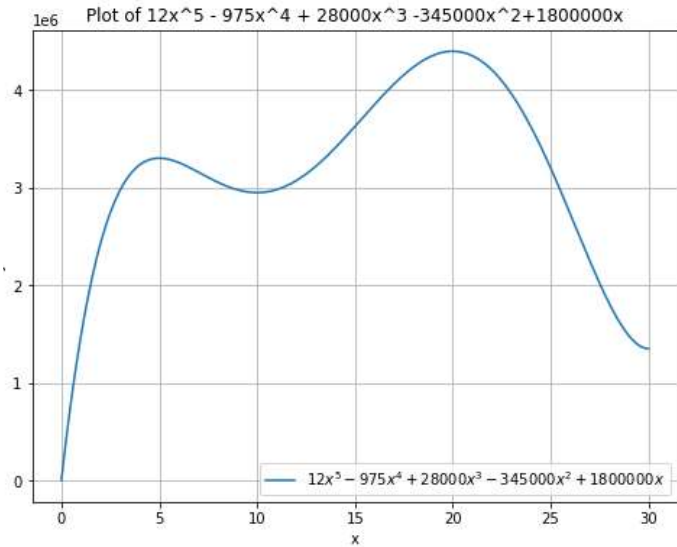
■ **TABLE 14.7** Application of the genetic algorithm to the integer nonlinear programming example through (a) the initialization step and (b) iteration 1

Member	Initial Population	Value of x	Fitness
1	0 1 1 1 1	15	3,628,125
2	0 0 1 0 0	4	3,234,688
3	0 1 0 0 0	8	3,055,616
4	1 0 1 1 1	23	3,962,091
5	0 1 0 1 0	10	2,950,000
6	0 1 0 0 1	9	2,978,613
7	0 0 1 0 1	5	3,303,125
8	1 0 0 1 0	18	4,239,216
9	1 1 1 1 0	30	1,350,000
10	1 0 1 0 1	21	4,353,187

(a)

Member	Parents	Children	Value of x	Fitness
10	1 0 1 0 1	0 0 1 0 1	5	3,303,125
2	0 0 1 0 0	1 0 0 0 1	17	4,064,259
8	1 0 0 1 0	1 0 0 1 1	19	4,357,164
4	1 0 1 1 1	1 0 1 0 0	20	4,400,000
1	0 1 1 1 1	0 1 0 1 1	11	2,980,637
6	0 1 0 0 1	0 1 1 1 1	15	3,628,125

(b)



All this is very nice to code; here the starting and first iteration from the Hillier book (using $x = \text{integer}$ for simplicity)



And the convergence is rapid for this simple case, good results already at the first iteration

The procedure can of course be applied to non-integer numbers, as these can as well be written in binary notation

Exercise: write 412 in binary starting from 256

1	0	0	0	0	0	0	0	0
256	128	64	32	16	8	4	2	1

I start by 256 because the next power of 2 (512) is too big; **get 1**

$412 - 256 = 156$; so I can fit in 128; **get 11**

The difference is 28, so I cannot fit a 64: **get 110**

Cannot fit a 32, **get 1100**

Can fit 16,8,4 add to 28 **get 1100111**

No need of 2 and 1 the last two powers, **get 110011100**



Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

256y 128y 64n 32n 16y 8y 4y 2n 1n

Faster algorithm

i) Divide the decimal number by 2 and write down the integer result and see if there is a remainder (0 or 1). (ii) Repeat step (i) with the integer result until you get 0. (iii) Write down the remainders (in reverse order) of each division to get the binary equivalent. We try again for 412

	remainder
$412/2=206$	0
$206/2=103$	0
$103/2=51$	1
$51/2=25$	1
$25/2=12$	1
$12/2=6$	0
$6/2=3$	0
$3/2=1$	1
$1/2=0$	1



Reading bottom up gives

110011100

Try with 3	$3/2=1$	1
	$1/2=0$	1 → 11

Try with 4	$4/2=2$	0
	$2/2=1$	0
	$1/2=0$	1 → 100

Try with 7	$7/2=3$	1
	$3/2=1$	1
	$1/2=0$	1 → 111

Try with 11	$11/2=5$	1
	$5/2=2$	1
	$2/2=1$	0
	$1/2=0$	1 → 1011

Hocus pocus? No, is the same with decimal numbers

Try with 18


$18/2=9$	0
$9/2=4$	1
$4/2=2$	0
$2/2=1$	0
$1/2=0$	$1 \rightarrow 10010$

Try with xviii

$xviii/10=1$	8
$1/10=0$	$1 \rightarrow 18$

Try with ccxvii

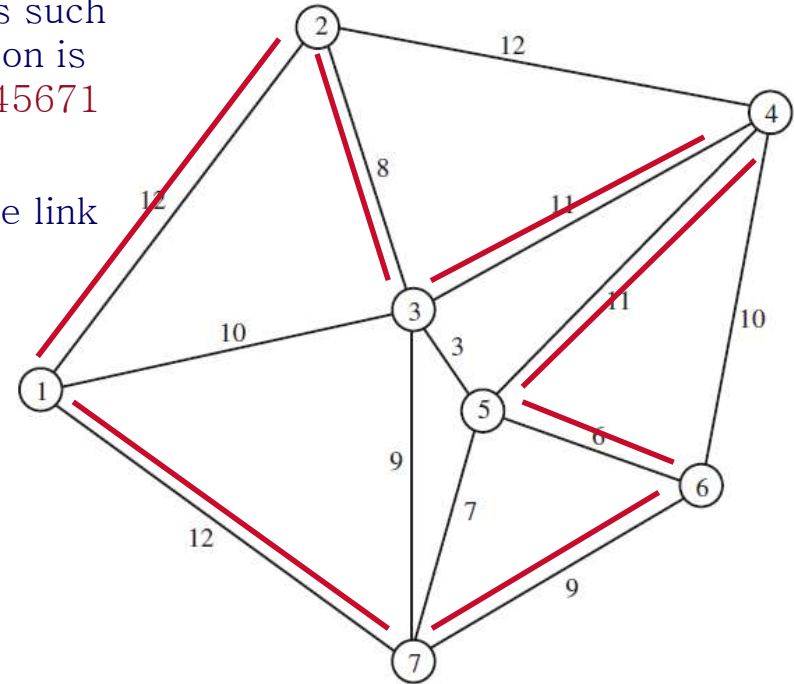
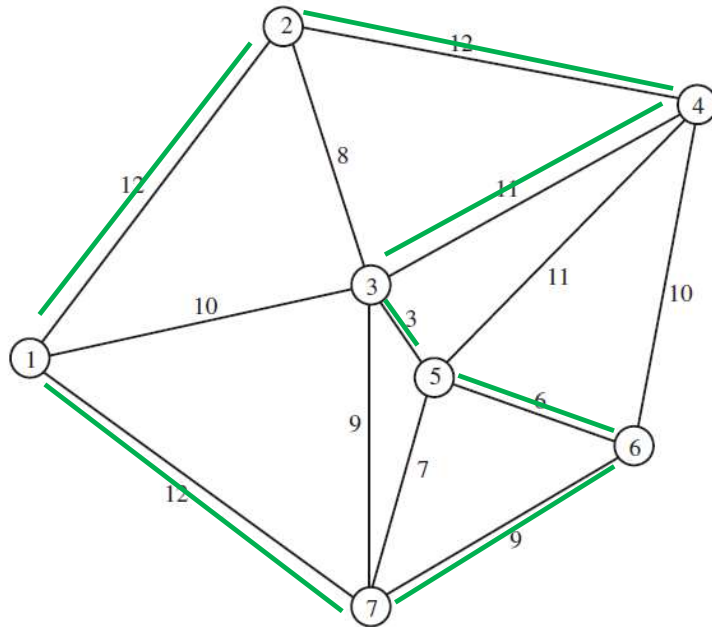
$ccxvii/10=21$	7
$21/10=2$	1
$2/10=0$	$2 \rightarrow 217$



We are used to look at this and say two hundreds one ten and seven units

Genetic algorithms can also be applied to problems such as the traveling salesman; in this case the population is constituted by candidate trajectories, such as 12345671 and 12435671

A child of this couple can inherit the link 2-3 or the link 2-4 ...



17.

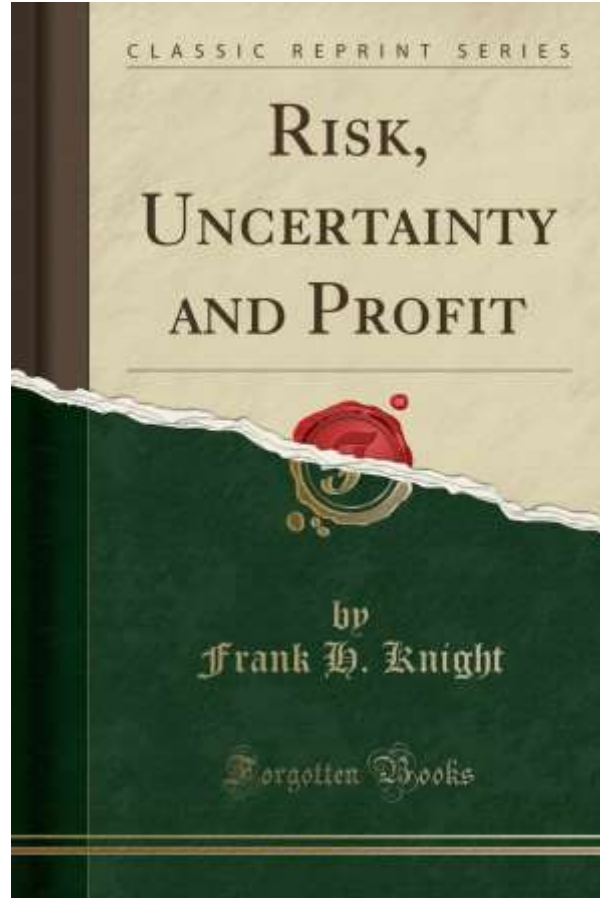
Decision Analysis

Knight. Decision making with and without experimentation. Example: drilling or selling? Bayes in full. Decision trees. Multi Criteria Decision Analysis. Linearization. Borda count, Condorcet's outranking matrix and Balinski-Laraki's majority judgment. Hillier (2014) chapter 16 plus various authors.

Frank Knight (1921) distinguished risk from uncertainty

Risk = know outcomes & probabilities;
roulette game

Uncertainty = unsure about the probabilities;
starting a business



Frank H. Knight
1885–1972

Quote:

“We live in a world of contradiction and paradox, a fact of which perhaps the most fundamental illustration is this: that the existence of a problem of knowledge depends on the future being different from the past, while the possibility of the solution of the problem depends on the future being like the past.”



Frank H. Knight
1885–1972

A prototype example of decision under uncertainty;
drilling or selling?



Source: <https://ecsgothermal.com/oil-drilling-on-land/>

A company own land
where there could be oil

Another company offers
to purchase said land



Source: <https://ecsgeothermal.com/oil-drilling-on-land/>

■ **TABLE 16.1** Prospective profits for the Goferbroke Company

Alternative \ Status of Land	Payoff	
	Oil	Dry
Drill for oil	\$700,000	−\$100,000
Sell the land	\$ 90,000	\$ 90,000
Chance of status	1 in 4	3 in 4

The table offers different payoffs associated to different decision (sell, drill) versus two possible states of nature (oil, no-oil)

■ **TABLE 16.1** Prospective profits for the Goferbroke Company

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Drill for oil	\$700,000	−\$100,000
Sell the land	\$ 90,000	\$ 90,000
Chance of status	1 in 4	3 in 4

How to act on this table? Different alternatives are available.

The Maximin Payoff Criterion

■ **TABLE 16.1** Prospective profits for the Goferbroke Company

Alternative \ Status of Land	Payoff	
	Oil	Dry
Drill for oil	\$700,000	−\$100,000
Sell the land	\$ 90,000	\$ 90,000
Chance of status	1 in 4	3 in 4

For each decision look at the worst payoff over all possible states of nature ...

...and choose the one with the best outcome

← **Drill:** Worse payoff for state of nature = dry, −100K

Sell: Worse payoff for state of nature? Indifferent (dry or oil the same payoff 90K)

← Sell, as 90 is better than −100

The Maximum likelihood approach

■ **TABLE 16.1** Prospective profits for the Goferbroke Company

Alternative \ Status of Land	Payoff	
	Oil	Dry
Drill for oil	\$700,000	−\$100,000
Sell the land	\$ 90,000	\$ 90,000
Chance of status	1 in 4	3 in 4

Identify the most likely state of nature

← Dry, as $\frac{3}{4}$ is more than $\frac{1}{4}$ (prior probabilities)

...and choose the one with the best pay-off

← Sell, as 90 is better than −100

The Maximum likelihood approach

■ **TABLE 16.4** Application of the maximum likelihood criterion to the first Goferbroke Co. problem

Alternative	State of Nature		
	Oil	Dry	
1. Drill for oil	700	-100	-100
2. Sell the land	90	90	90 ← Maximum in this column
Prior probability	0.25	0.75	
		↑	
		Maximum	

Bayes' rule – the expected value approach

Dig!

Alternative	State of Nature	
	Oil	Dry
1. Drill for oil	700	-100
2. Sell the land	90	90
Prior probability	0.25	0.75



Calculate the best expected
payoff for each decision
alternative

$$\text{Payoff (Drill)} = 0.25 \times 700 - 0.75 \times 100 = 100$$

$$\text{Payoff (Sell)} = 0.25 \times 90 - 0.75 \times 90 = 90$$

Before we proceed with Reverend Bayes, remember the caveat of lesson one: expected value may lead to counter intuitive results



◦ This is the story the St. Petersburg paradox (another game!)

Would you accept one million dollars with certainty or one chance in ten of winning 20 millions?



Daniel Bernoulli
(1700-1782)

Bayes' rule – the expected value approach

Dig!

Alternative	State of Nature	
	Oil	Dry
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Prior probability	0.25	0.75



What do we do if we feel uneasy with these prior probabilities? What if instead of 0.25 the probability of oil is instead 0.15 or 0.35?

	State of Nature	
	Oil	Dry
1. Drill for oil	700	−100
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Prior probability	0.25	0.75

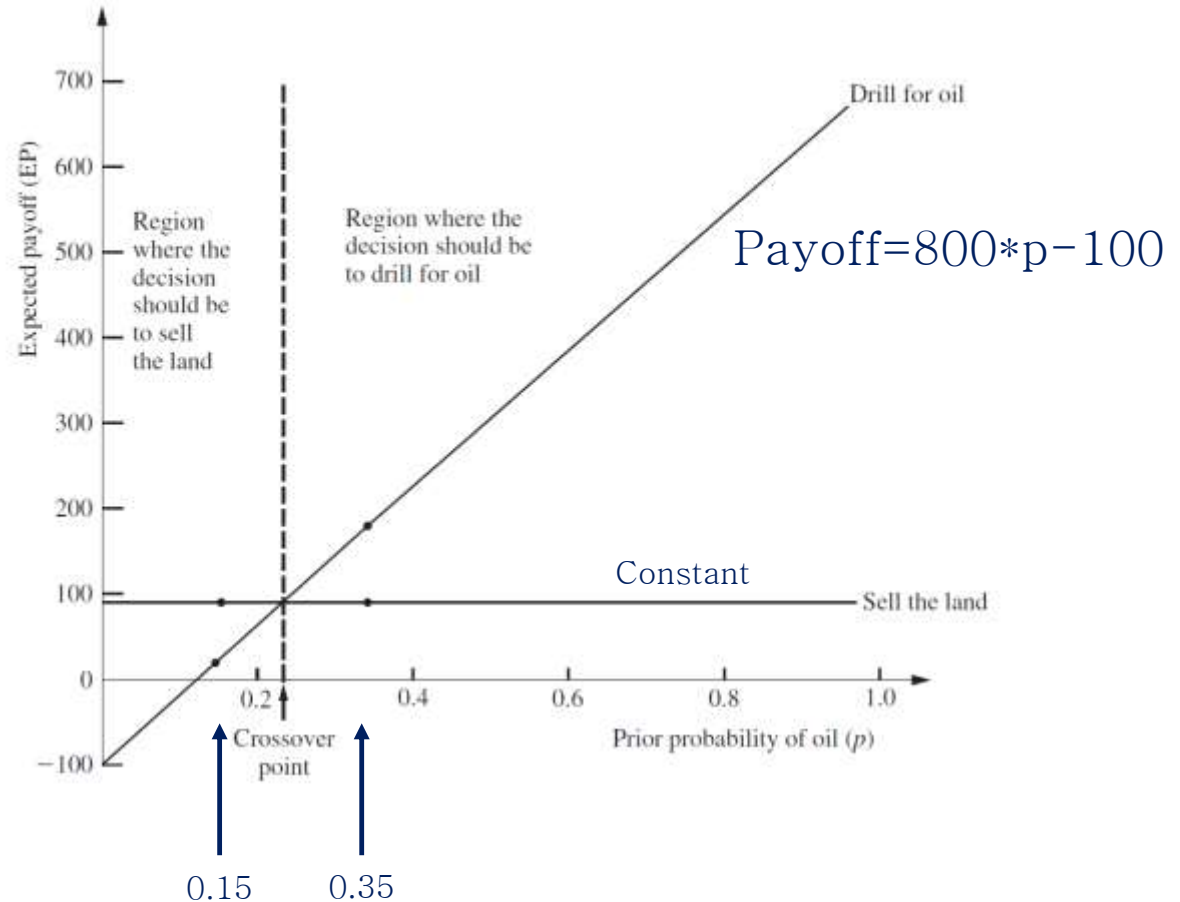
What do we do if we feel uneasy with these prior probabilities? What if instead of 0.25 the probability of oil is instead 0.15 or 0.35?

The payoff for a generic value p of this prior is

$$p \cdot 700 - (1-p) \cdot 100 = 800 \cdot p - 100$$

Alternative	State of Nature	
	Oil	Dry
1. Drill for oil	700	-100
2. Sell the land	90	90
Prior probability	0.25	0.75

↑
 What if this is
 not 0.25, but
 for example
 0.15 or 0.35?

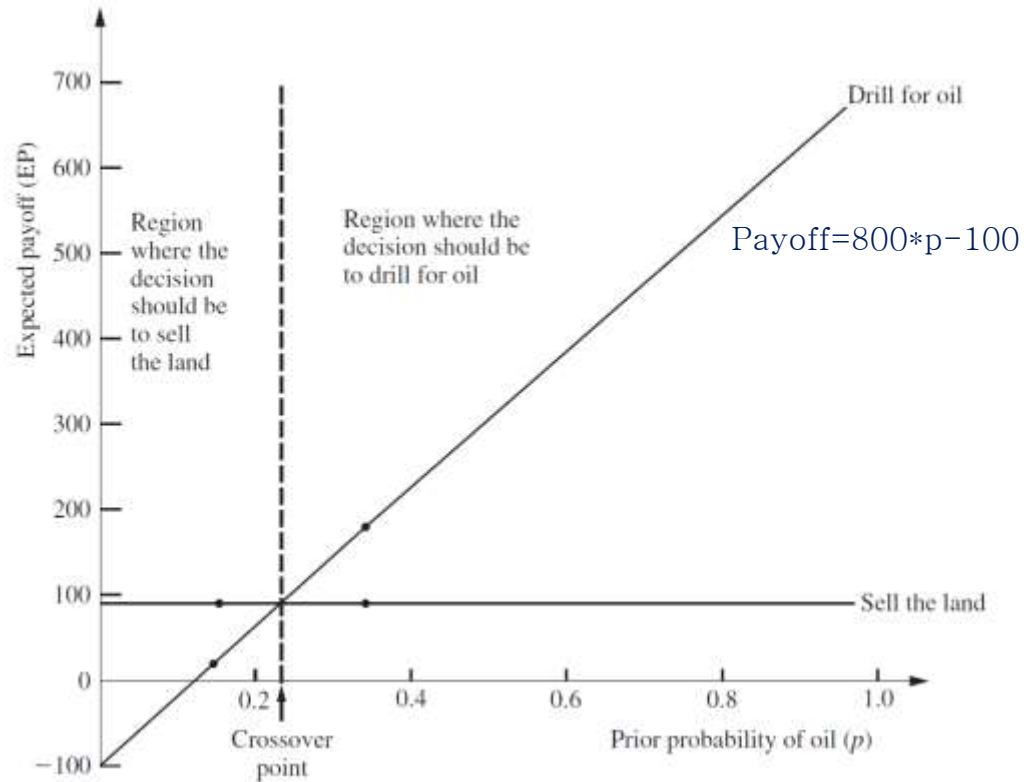


Alternative	State of Nature	
	Oil	Dry
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Prior probability	0.25	0.75

Exercise: compute cross over coordinates



Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous



Alternative	State of Nature	
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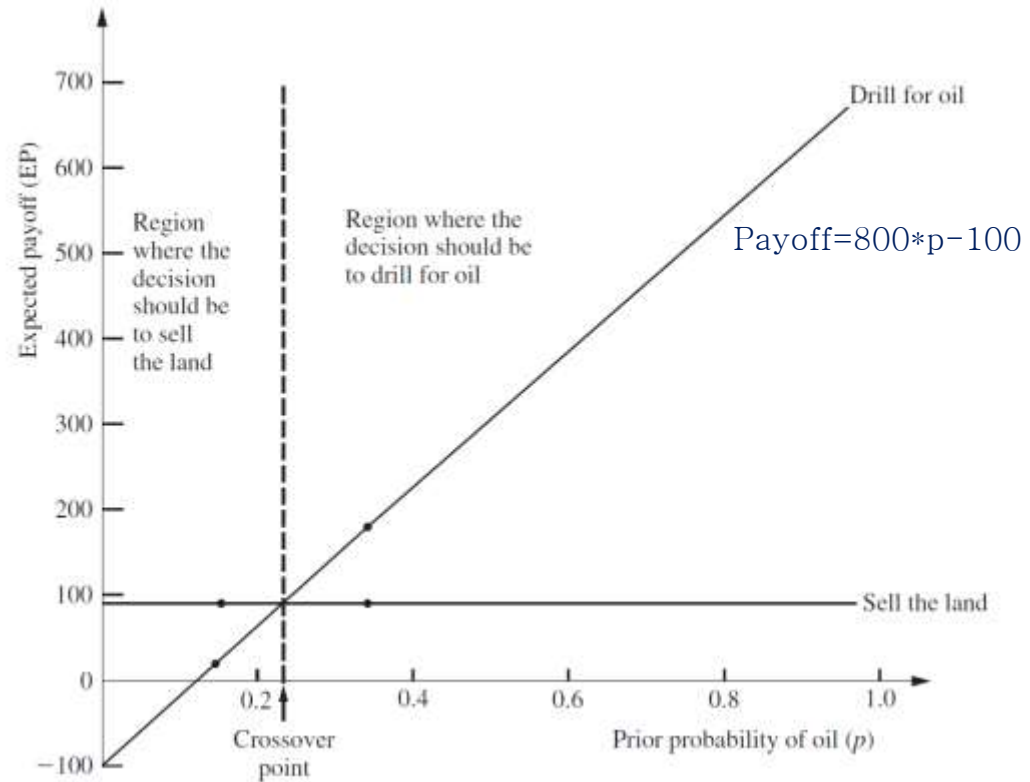
Exercise: compute cross over coordinates

Intersection of
 $y = 800x - 100$

and

$$y = 90$$

$$x = 190/800 = .2375$$

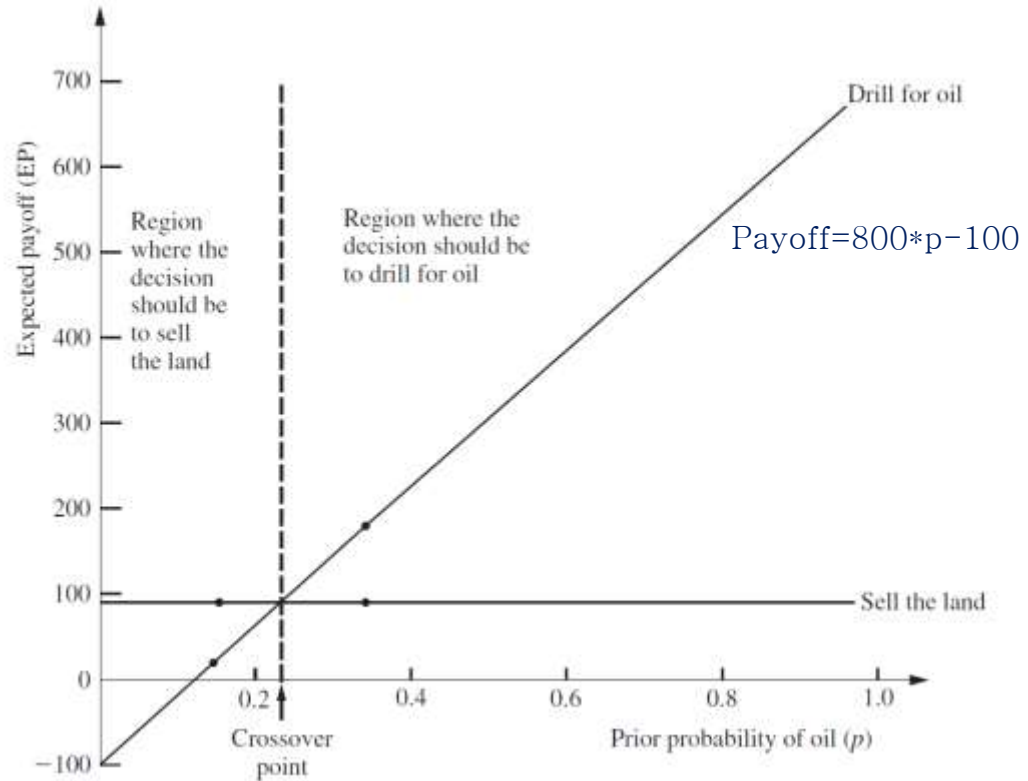


Alternative	State of Nature	
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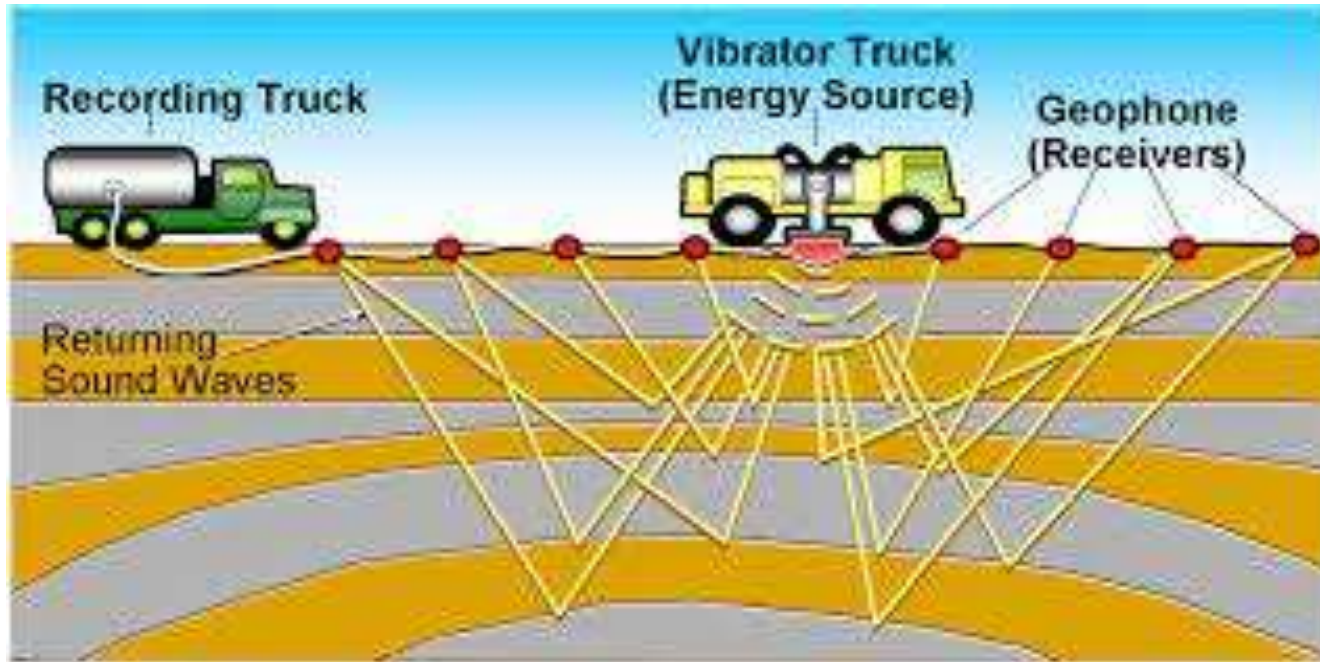
Conclusion:

if $p < .2375$ then sell

if $p > .2375$ then drill



Decision making with experimentation



Source: https://gov.nu.ca/sites/default/files/2017_seismic_eng.pdf

Perhaps before deciding whether to sell or drill some prospection study should be done, such as seismic surveying

This would come to a cost, so even in this case, before the survey, it would be wise to crunch some numbers

The cost of the seismic survey is \$30,000.



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#8669

service@toonclips.com

Experience says that:

USS: Unfavorable Seismic Soundings → oil is fairly unlikely.

FSS: Favorable Seismic Soundings → oil is fairly likely.

Again experience translates this into

(these are data of the problem; they are given to us)

$$p(USS|Oil) = 0.4 \text{ and } p(FSS|Oil) = 1 - 0.4 = 0.6$$

$$p(USS|Dry) = 0.8 \text{ and } p(FSS|Dry) = 1 - 0.8 = 0.2$$

In plain English:

Geologists tell us that **if the oil is there**, the test will be negative (no oil) 40% of the times and positive (yes there is oil) 60% of the times

... and that **if the oil is not there** the test will be negative 80% of the times and positive 20% of the times

As it is written, the famous theorem ‘looks’ symmetric in A and B ...



$$p(A|B)p(B) = p(B|A)p(A) = p(A \cap B)$$



In fact the way it is used in practice is rather asymmetric, and aims to update A based on B being true, B being for example an experiment and A a theory

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

When B is the outcome of an experiment and A is a state of nature $p(A|B)$ becomes the probability that given the outcome B (for example a favourable outcome FSS) then we indeed have A – the oil in this case;

we do not know $p(A|B)$ but we do know $p(B|A)$, in this case the probability that if there is oil the test will be favourable

$$p(Oil|FSS)P(FSS) = p(FSS|Oil)P(Oil)$$



we do not know $p(A|B)$ but we do know $p(B|A)$, in this case the probability that if there is oil the test will be favourable

We also know $P(Oil)$, as this is the old prior, the probability of oil being there before the survey

$$p(Oil|FSS)P(FSS) = p(FSS|Oil)P(Oil)$$



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$$p(Oil|FSS)P(FSS) = p(FSS|Oil)P(Oil)$$

We only lack $P(FSS)$. This is a delicate point. The unconditional probability of favourable drilling is the total probability of this outcome in all cases, e.g. both if there is oil and if there is no-oil



$$p(Oil|FSS)P(FSS) = p(FSS|Oil)P(Oil)$$

We only lack $P(FSS)$. This is a delicate point. The unconditional probability of favourable drilling is the total probability of this outcome in all cases, e.g. both oil and no-oil



How about:

$$P(FSS) = p(FSS|Oil)P(Oil) + p(FSS|No - oil)P(No - oil)$$

This is indeed the total, and hence unconditional, probability of FSS – that is to say all possible ways in which FSS can come about

Putting these two together:

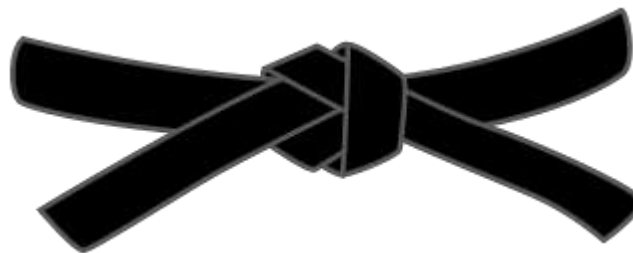
$$p(Oil|FSS)P(FSS) = p(FSS|Oil)P(Oil)$$

$$P(FSS) = p(FSS|Oil)P(Oil) + p(FSS|No - oil)P(No - oil)$$

we get

$$p(Oil|FSS) = \frac{p(FSS|Oil)P(Oil)}{p(FSS|Oil)P(Oil) + p(FSS|No - oil)P(No - oil)}$$

You have just done your
first Bayesian updating



Plugging the numbers

$$p(Oil|FSS) = \frac{\overset{0.6}{p(FSS|Oil)}\overset{0.25}{P(Oil)}}{\underset{0.6}{p(FSS|Oil)}\underset{0.25}{P(Oil)} + \underset{0.20}{p(FSS|No - oil)}\underset{0.75}{P(No - oil)}}$$

=0.15/(.15+.15)=0.5

And this gives $p(Oil|FSS) = \frac{1}{2}$



Hence since

$$p(Oil|FSS) = \frac{1}{2}$$

then

$$p(Dry|FSS) = \frac{1}{2}$$

And following a similar path for the negative survey outcome *USS*

$$p(Oil|USS) = \frac{1}{7} = 1.4$$

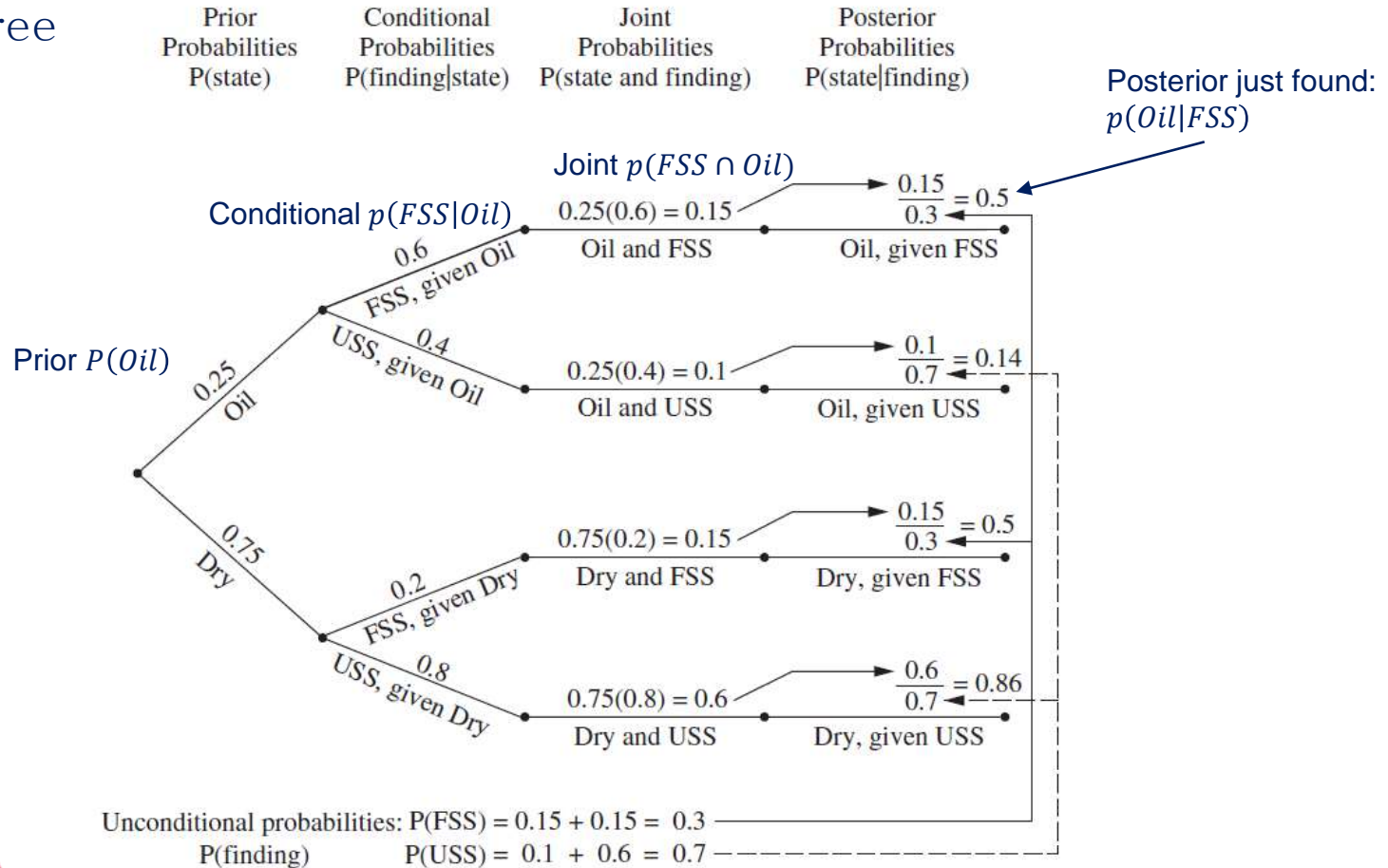
$$p(Dry|USS) = \frac{6}{7} = .86$$



Indeed the survey is a game changer when compared to the prior probabilities $P(Oil)=0.25$ and $P(Dry) = 0.75$

- ➔ Probably nobody would be a taker for drill if *USS* is true
- ➔ One half is much better than one in four if *FSS* is true

All in a tree



We now need to use all these

$$p(Oil|FSS) = \frac{1}{2}$$

$$p(Dry|FSS) = \frac{1}{2}$$

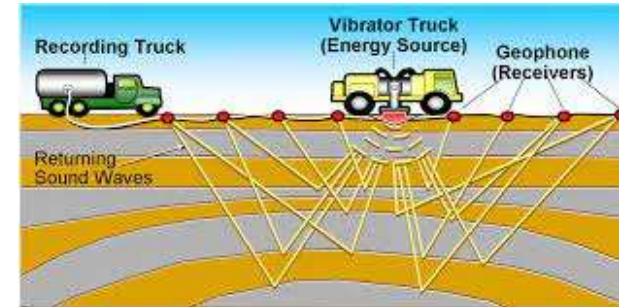
$$p(Oil|USS) = \frac{1}{7}$$

$$p(Dry|USS) = \frac{6}{7}$$

to take a decision, about drill, sell, and survey



Source: <https://ecsgeothermal.com/oil-drilling-on-land/>



Source: https://gov.nu.ca/sites/default/files/2017_seismic_eng.pdf

This is now straightforward:

Payoffs if unfavourable survey (*USS*):

$$E(\text{Payoff} - \text{Drill} | \text{USS}) = \frac{1}{7}(700) + \frac{6}{7}(-100) - 30 = -15.7$$

$$E(\text{Payoff} - \text{Sell} | \text{USS}) = \frac{1}{7}(90) + \frac{6}{7}(90) - 30 = 60$$

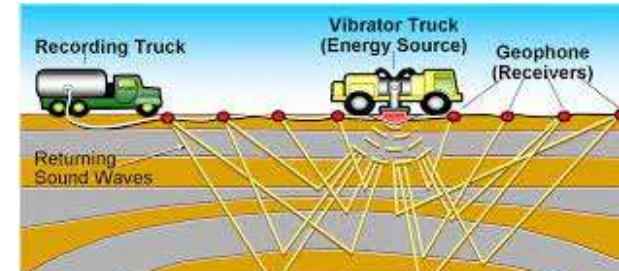
Payoffs if favourable survey (*FSS*):

$$E(\text{Payoff} - \text{Drill} | \text{FSS}) = \frac{1}{2}(700) + \frac{1}{2}(-100) - 30 = 270$$

$$E(\text{Payoff} - \text{Sell} | \text{FSS}) = \frac{1}{2}(90) + \frac{1}{2}(90) - 30 = 60$$

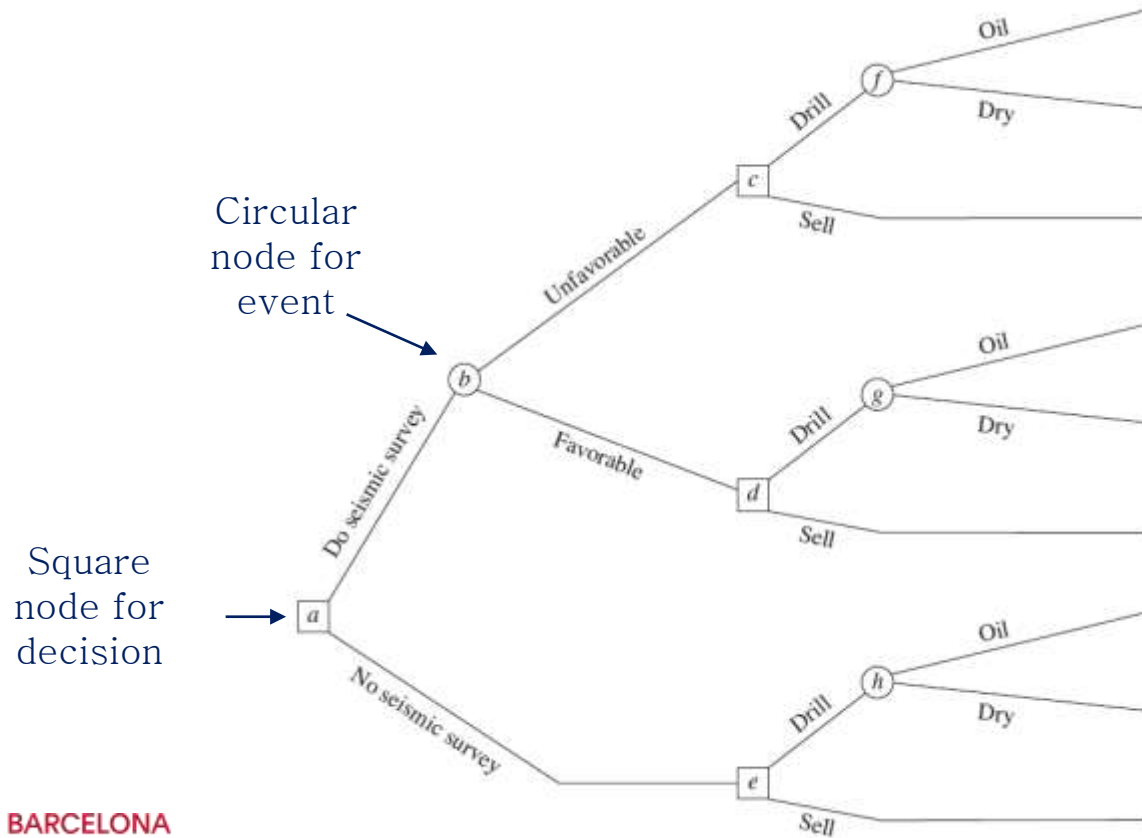


Source: <https://ecsgeothermal.com/oil-drilling-on-land/>



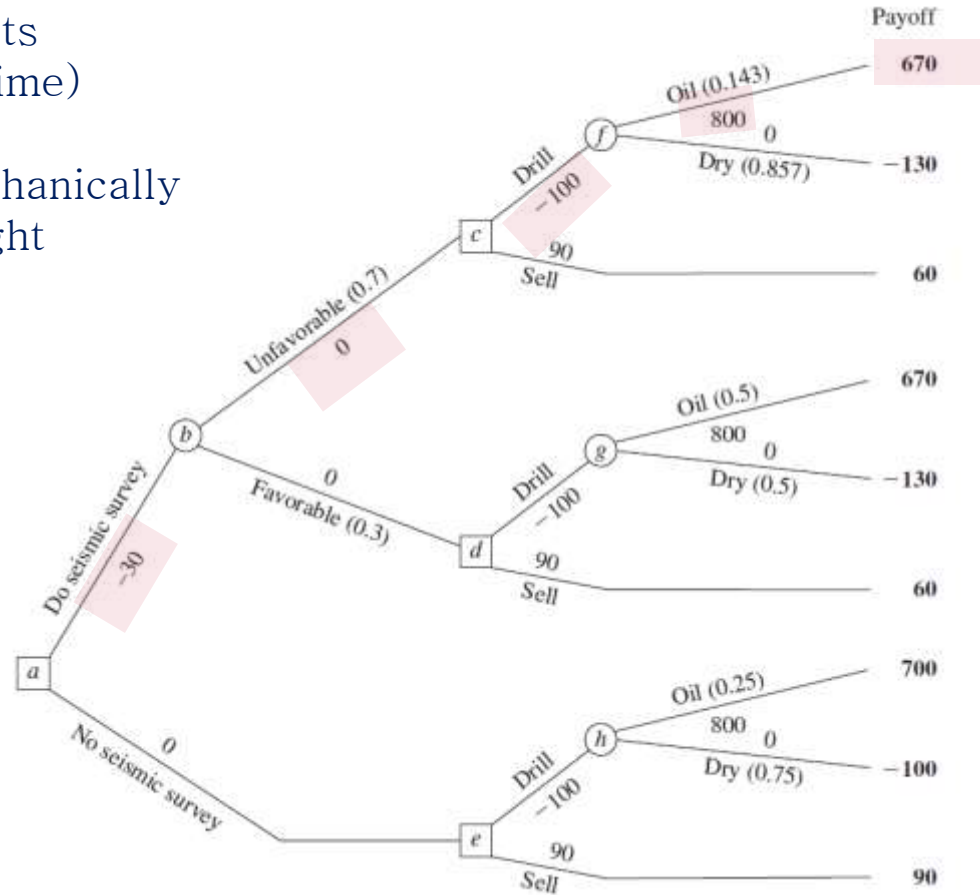
Alternative	State of Nature	
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Decision tree for the same problem



Decision tree with costs (no probabilities this time)

Cost are compute mechanically
moving from left to right



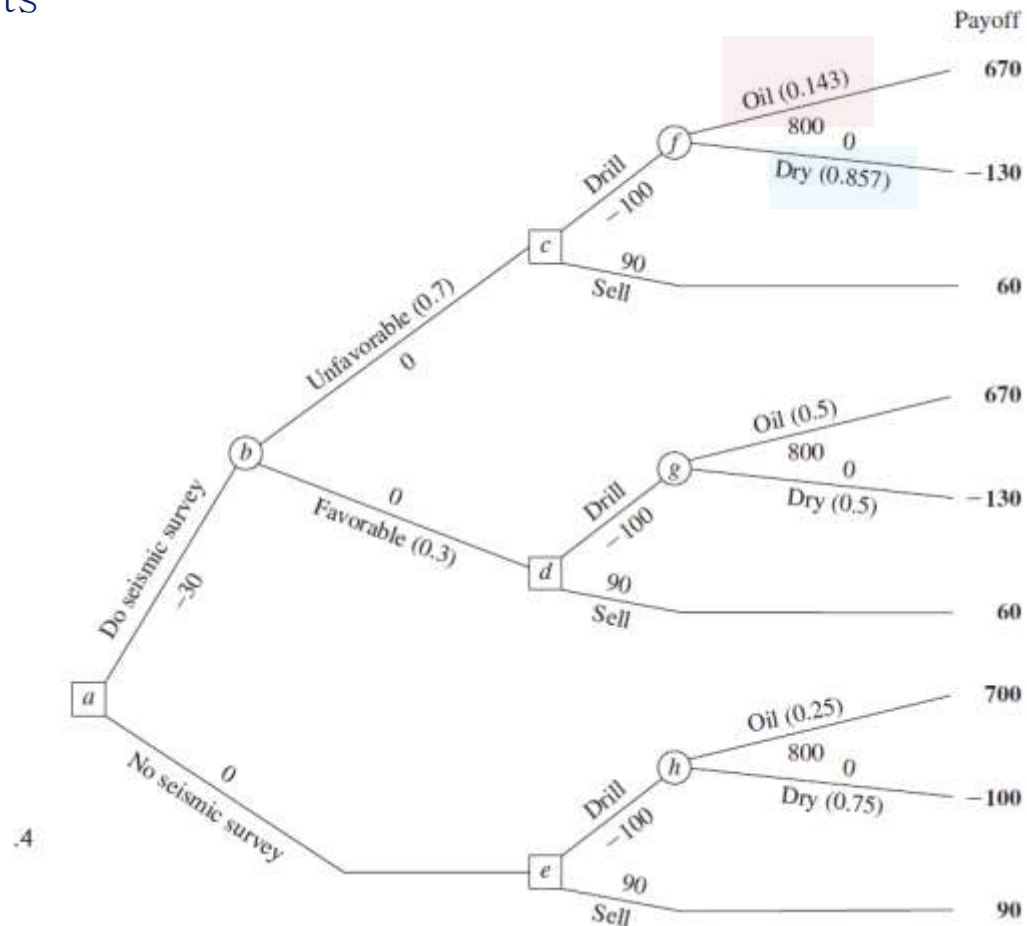
Decision tree with costs (adding probabilities)

$$p(Oil|FSS) = \frac{1}{2} = 0.5$$

$$p(Dry|FSS) = \frac{1}{2} = 0.5$$

$$p(Oil|USS) = \frac{1}{7} = 0.143$$

$$p(Dry|USS) = \frac{6}{7} = 0.857$$



Transforming this into a decision tree. Recipe: from the rightmost column look left, then

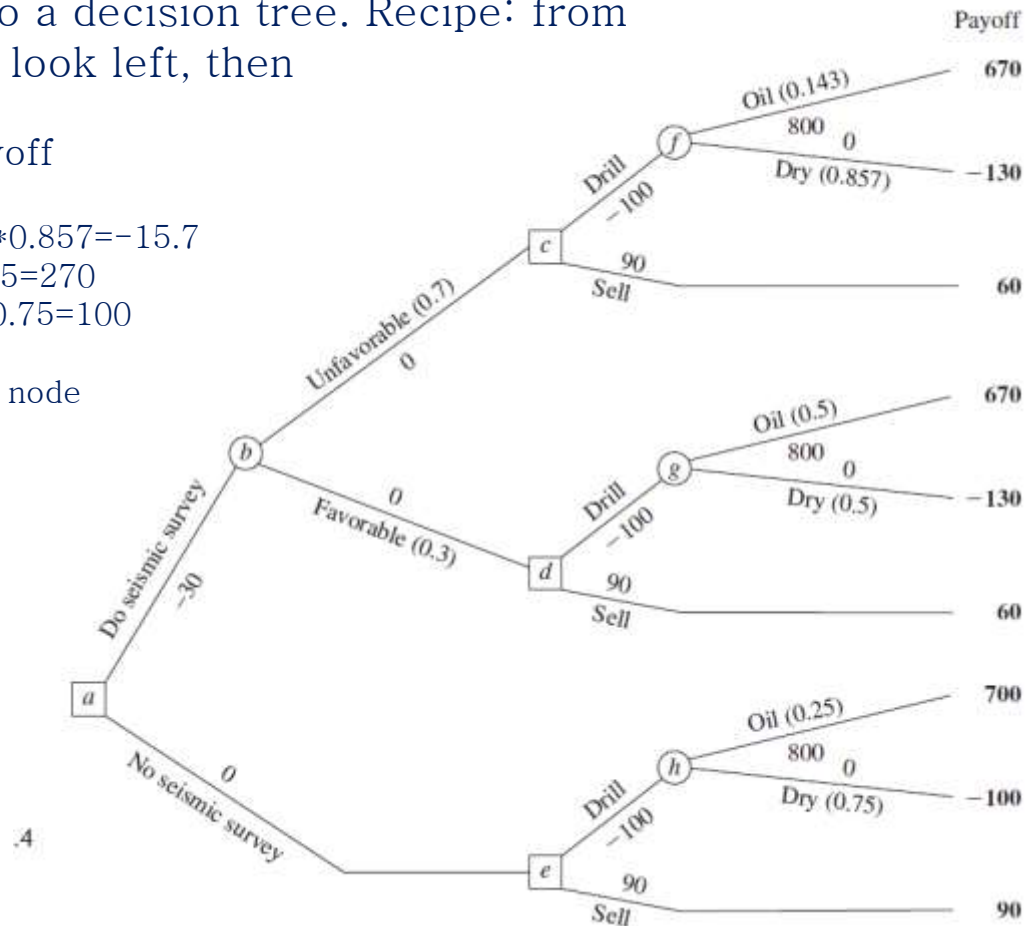
If node=event compute payoff

$$\text{Payoff}(f) = 670 \cdot 0.143 + (-130) \cdot 0.857 = -15.7$$

$$\text{Payoff}(g) = 670 \cdot 0.5 + (-130) \cdot 0.5 = 270$$

$$\text{Payoff}(h) = 700 \cdot 0.25 + (-100) \cdot 0.75 = 100$$

Write these numbers above the node



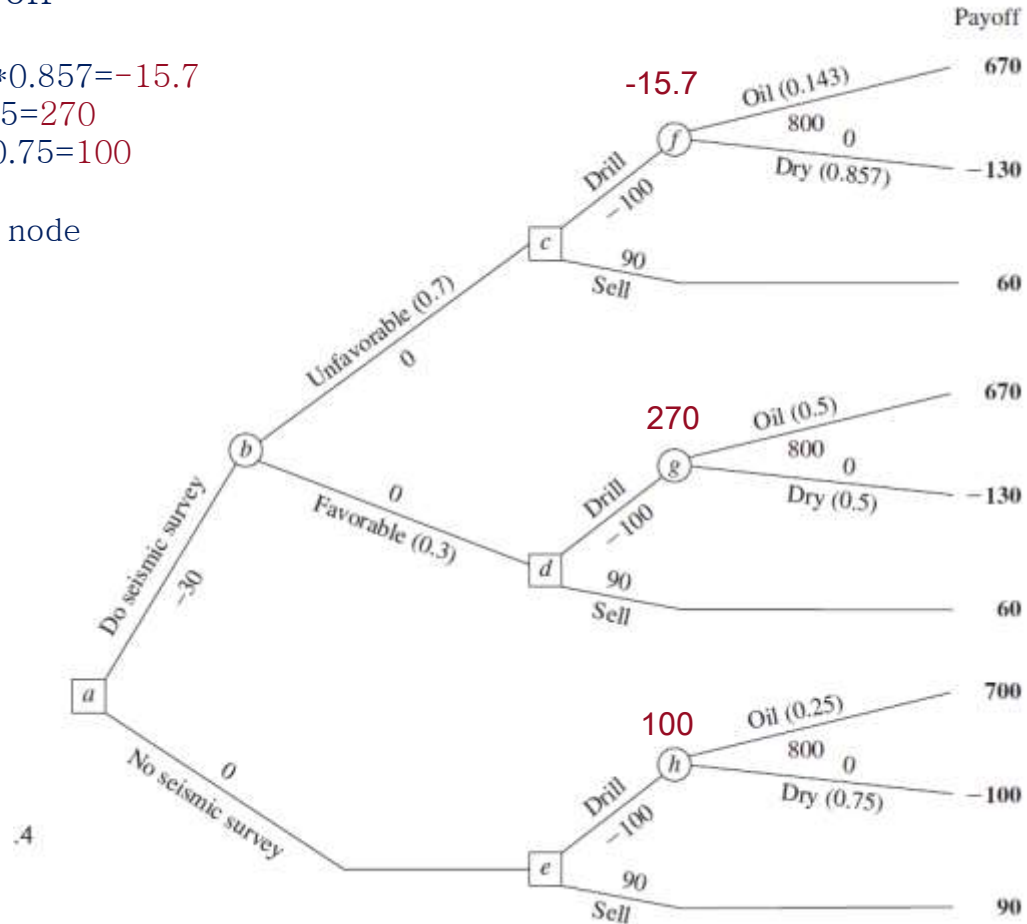
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$$\text{Payoff}(h) = 700 \cdot 0.25 + (-100) \cdot 0.75 = 100$$

Write these numbers above the node

If node=decision then decide

Decision(c) = Sell

Decision(d) = Drill

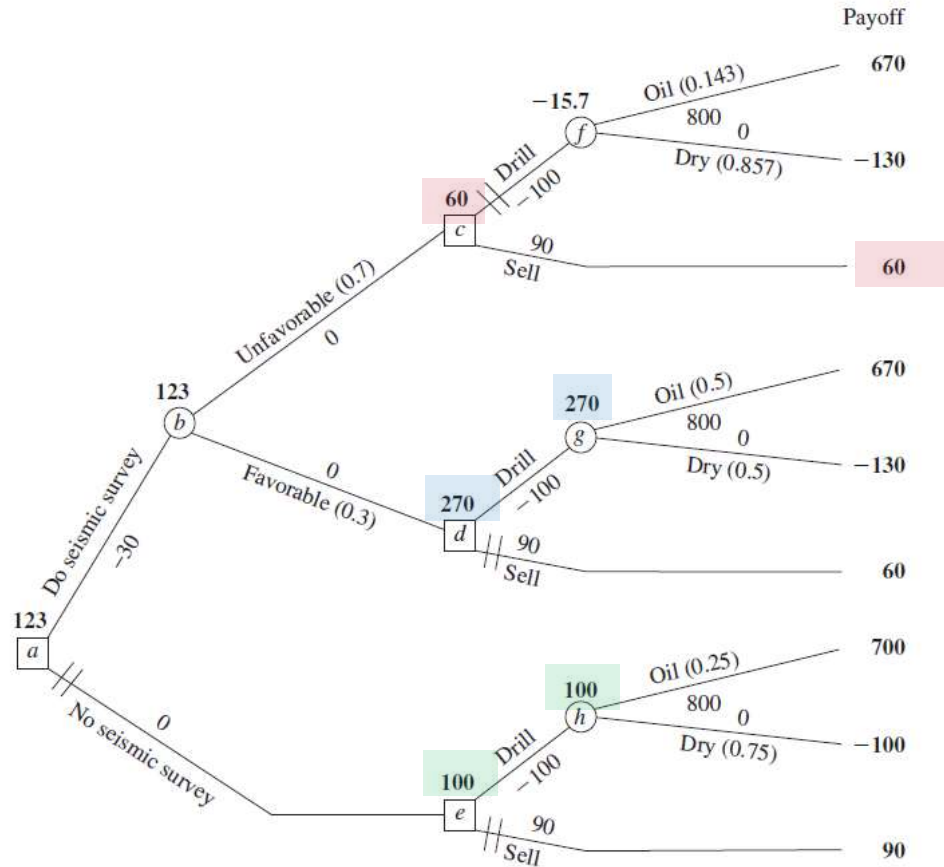
Decision(e) = Drill

Report the payoff selected above the node

Move left

$$\text{Payoff}(b) = 60 \cdot 0.7 + 270 \cdot 0.3 = 123$$

Decision(a) = Do survey



How to deal with possible paradoxes when using expected value (our old slide again):

- This is the story the St. Petersburg paradox (another game!)

Would you accept one million dollars with certainty or one chance in ten of winning 20 millions?



Daniel Bernoulli
(1700-1782)

→ Utility theory

- This is the story the St. Petersburg paradox (another game!)

Would you accept one million dollars with certainty or one chance in ten of winning 20 millions?



Daniel Bernoulli
(1700-1782)

→ When using Utility Theory the rhetorical question above becomes the tool to elicit users preferences

→ When using Utility Theory the rhetorical question above becomes the tool to elicit users preferences

A common occurrence if that actors show a decreasing marginal utility for money (risk aversion)

To see if this is the case and to elicit the values for the utilities, the following alternatives are posed to the actor

Receiving \$10,000 with certainty

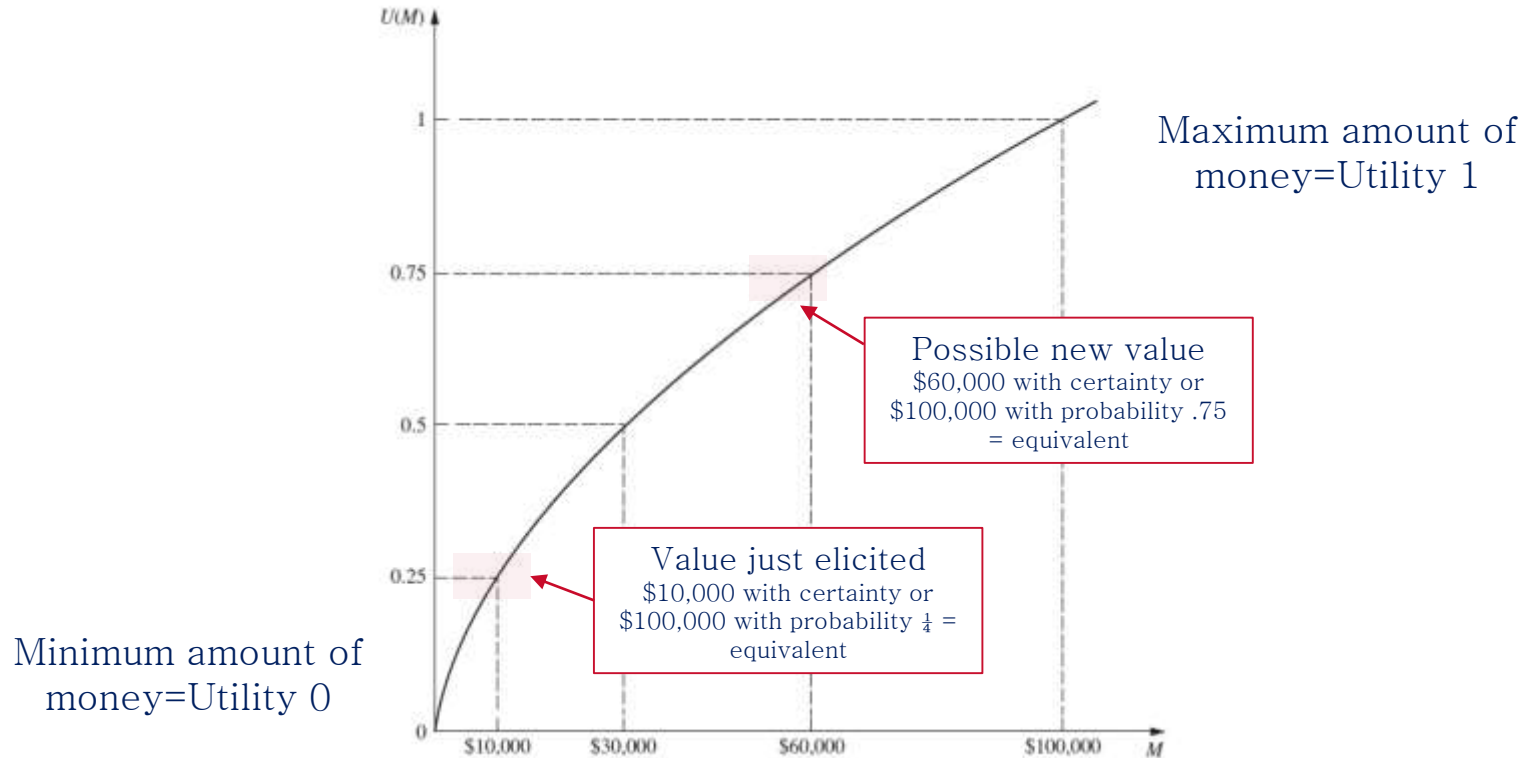
Receiving 100,000 with probability p

To see if this is the case and to elicit the values for the utilities, the following alternatives are posed to the actor

- 1) Receiving \$10,000 with certainty
- 2) Receiving \$100,000 with probability p (\$100,000 is the upper limit of the curve we intend to build)

The following question is posed: for what value of p would you consider options 1 and 2 equivalent. Imagine the answer is $\frac{1}{4}$ ($p = 0.25$) \rightarrow the actor consider \$10,000 with certainty and \$100,000 with probability $\frac{1}{4}$ as equivalent

\$10,000 with certainty or \$100,000 with probability $\frac{1}{4}$ = equivalent

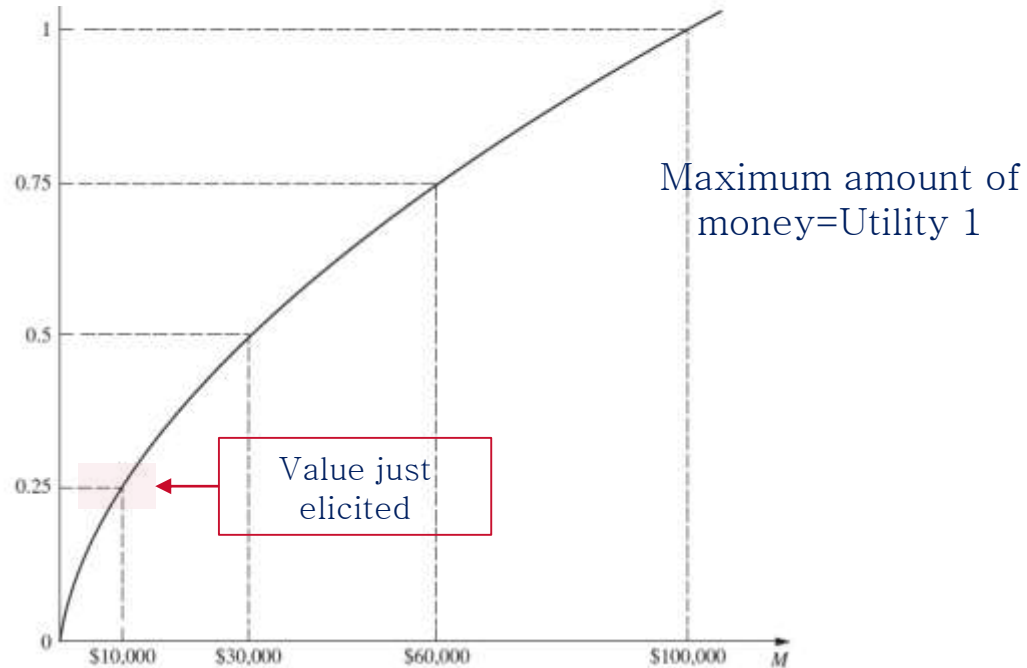


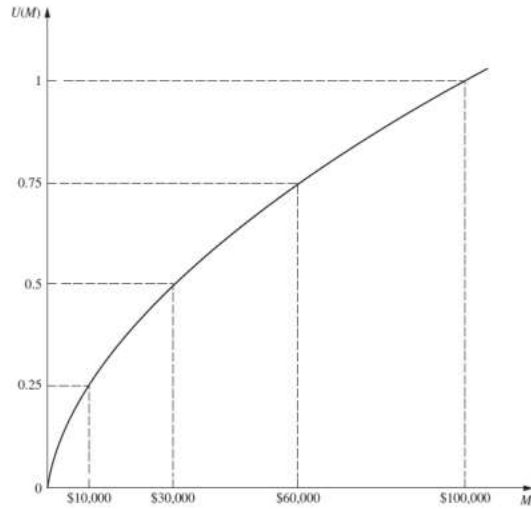
- 1) Receiving \$10,000 with certainty
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The following question is posed: for what value of p would you consider options 1 and 2 equivalent. Imagine the answer is $\frac{1}{4}$ ($p = 0.25$) \rightarrow the actor consider \$10,000 with certainty of \$100,000 with probability $\frac{1}{4}$ as equivalent

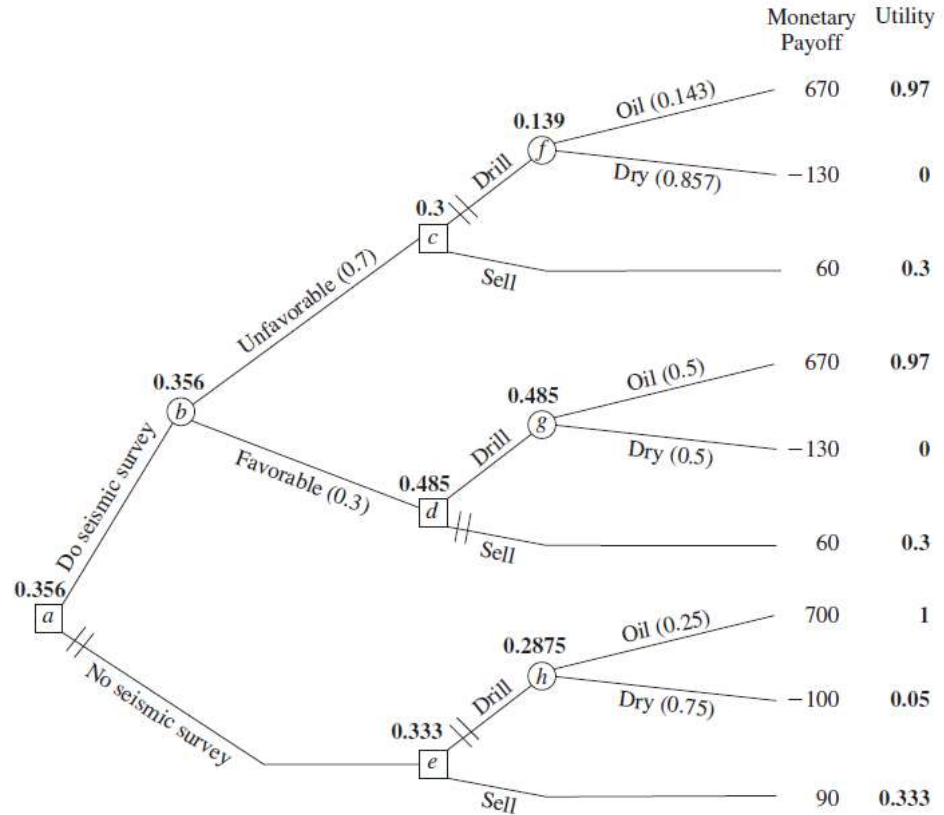
Repeating this for values different than \$10,000
The utility curve can be built and used in decision analysis, simply replacing monetary payoff with utilities

Minimum amount of money=Utility 0





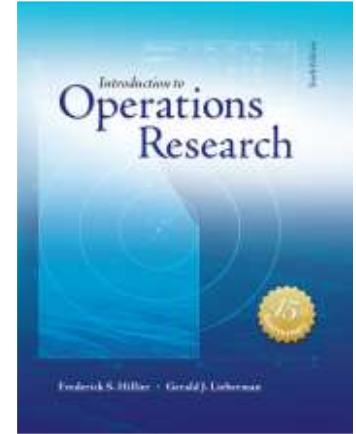
Using the utility curve the monetary payoff is replaced with utilities

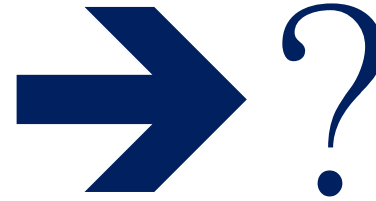


“However, many decision makers are not sufficiently comfortable with the relatively abstract notion of utilities, or with working with probabilities to construct a utility function, to be willing to use this approach. Consequently, utility theory is not yet used very widely in practice” (p. 715)



Source: <https://www.alamy.com/>





Source: <https://www.alamy.com/>

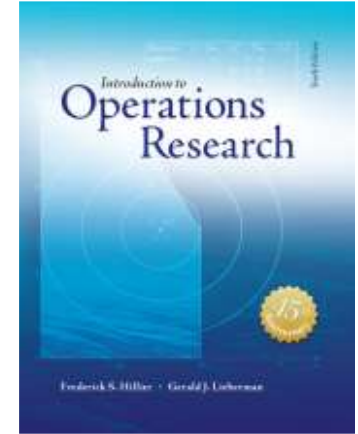
This idiosyncrasy to reckon in terms of abstract utilities or probabilities needs to be kept in mind if decision are taken in teams, e.g. in *Decision Conferencing*

Multiple criteria decision analysis

Using simultaneously more than one criterion

E.g. a company wishing to meet simultaneously goals of

- Profit
- Employment
- Capital investments



A company wishing to meet simultaneously goals of

- Profit ≥ 125 (millions of dollars)
- Employment = 4 (hundreds of employees)
- Capital investments ≤ 55 investment goal

in the commercialization of three products (decision variables) x_1, x_2, x_3
Goals can be one sided upper (capital investment) or lower (profit) or two sided (employment).

The relation between decision variables and goals is defined as:

$$12x_1 + 9x_2 + 15x_3 \geq 125$$

$$5x_1 + 3x_2 + 4x_3 = 40$$

$$5x_1 + 7x_2 + 8x_3 \leq 55$$

The relation between decision variables and goals is defined as:

$$12x_1 + 9x_2 + 15x_3 \geq 125$$

$$5x_1 + 3x_2 + 4x_3 = 40$$

$$5x_1 + 7x_2 + 8x_3 \leq 55$$

Note: MCDA section and this example are not available in the online version; this comes for the 11th version



A penalty weight is attached to violating the goal, i.e.

Weight=5 per unit below profit goal

Weight=3 per unit over investment goal

Weight=4 per unit over employment goal

Weight=2 per unit below employment goal

So the problem is linearized as

Minimize $Z = 5(\text{amount under profit goal}) + 3(\text{amount over investment goal}) + 4(\text{amount over employment goal}) + 2(\text{amount below employment goal})$

So the problem is linearized as

Minimize $Z = 5(\text{amount under profit goal}) + 3(\text{amount over investment goal}) + 4(\text{amount over employment goal}) + 2(\text{amount below employment goal})$

Extreme caution should be used in this kind of linearization, as the use of penalty weights in a linear model may lead to paradoxes – we just ‘scratch’ the problem here and suggest to use different approaches



So the problem is linearized as

Minimize $Z = 5(\text{amount under profit goal}) + 3(\text{amount over investment goal}) + 4(\text{amount over employment goal}) + 2(\text{amount below employment goal})$

Extreme caution should be used in this kind of linearization, as the use of penalty weights in a linear form may lead to paradoxes – e.g. when the items above have appreciable covariance

Suggestion: list different viable options and rank them using methods such as Borda, Condorcet, Balinski-Laraki ...



SOCIAL CHOICE AND
MULTICRITERION DECISION-MAKING

KENNETH J. ARROW AND HERVE RAYNAUD

AN MIT PRESS CLASSIC

MAJORITY
JUDGMENT

Measuring, Ranking, and Electing



MICHEL BALINSKI AND RIDA LARAKI



Some of these methods have a long history
(including in Catalonia)



Ramon Llull (Catalan, ca. 1232 – ca. 1315) proposed first what would then become known as the method of Condorcet. **Nicholas of Kues** (1401 – August 11, 1464), also referred to as Nicolaus Cusanus and Nicholas of Cusa developed what would later be known as the method of Borda. **Nicolas de Condorcet**, (17 September 1743 – 28 March 1794) developed the eponymous method. **Jean-Charles, chevalier de Borda** (May 4, 1733 – February 19, 1799) developed the Borda count

Images from Wikipedia Commons

An impact matrix

		↑	↓	↓	↓	↓
	Indic.	GDP	Unemp. Rate	Solid wastes	Income dispar.	Crime rate
Country						
A		25,000	0.15	0.4	9.2	40
B		45,000	0.10	0.7	13.2	52
C		20,000	0.08	0.35	5.3	80
weights		.166	.166	0.333	.166	.166

We can say that

GDP 'votes' for B>A>C (countries / options)

UR 'votes' for C>B>A

SW 'votes' for C>B>A

ID 'votes' for C>A>B

CR 'votes' for A>B>C

	Indic.	GDP	Unemp. Rate	Solid wastes	Income dispar.	Crime rate
Country						
A		25,000	0.15	0.4	9.2	40
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C		20,000	0.08	0.35	5.3	80
weights		.166	.166	0.333	.166	.166

# of indicators	2	1	1	1
1st position	<i>c</i>	<i>b</i>	<i>c</i>	<i>a</i>
2nd position	<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>
3rd position	<i>b</i>	<i>c</i>	<i>a</i>	<i>c</i>

GDP: B>A>C
 UR: C>B>A
 SW: C>A>B
 ID: C>A>B
 CR: A>B>C

# of indicators	2	1	1	1
1st position	<i>c</i>	<i>b</i>	<i>c</i>	<i>a</i>
2nd position	<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>
3rd position	<i>b</i>	<i>c</i>	<i>a</i>	<i>c</i>



Rank	<i>a</i>	<i>b</i>	<i>c</i>
1st	1	1	3
2nd	3	2	0
3rd	1	2	2

Different ways to organize the same information: building a frequency matrix

Three countries [options/candidates] and five indicators [criteria/voters]

# of indicators	2	1	1	1
1st position	<i>c</i>	<i>b</i>	<i>c</i>	<i>a</i>
2nd position	<i>a</i>	<i>a</i>	<i>b</i>	<i>b</i>
3rd position	<i>b</i>	<i>c</i>	<i>a</i>	<i>c</i>



Rank	<i>a</i>	<i>b</i>	<i>c</i>
1st	1	1	3
2nd	3	2	0
3rd	1	2	2

In this case Borda gives 3 minus 1 for each first rank , 2 minus 1 for each second rank and zero to the third

a gets $2*1 + 1*3 = 5$

b gets $2*1 + 1*2 = 4$

c gets $2*3 + 1*0 = 6$

But lets try Borda on a more interesting case:
(from Moulin, 21 criteria 4 options, cited in Munda 2008)



21 criteria 4 alternatives

Note: $3 + 5 + 7 + 6 = 21$

# of indicators	3	5	7	6
1st position	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>
2nd position	<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>
3rd position	<i>c</i>	<i>b</i>	<i>c</i>	<i>d</i>
4th position	<i>d</i>	<i>d</i>	<i>a</i>	<i>a</i>



Rank	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	Points
1st	8	7	6	0	3
2nd	0	9	5	7	2
3rd	0	5	10	6	1
4th	13	0	0	8	0

Borda count – Frequency matrix (Moulin, 21 criteria 4 options)

Columns add up to the
number of criteria /
voters=21

3 points if first
2 if second
1 if third
0 if last

Rank	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	Points
1st	8	7	6	0	3
2nd	0	9	5	7	2
3rd	0	5	10	6	1
4th	13	0	0	8	0

Borda score:

$$a = 8 \times 3 = 24$$

$$b = 5 + 9 \times 2 + 7 \times 3 = 44$$

$$c = 10 + 5 \times 2 + 6 \times 3 = 38$$

$$d = 6 + 7 \times 2 = 20$$

Borda solution:

$b \rightarrow c \rightarrow a \rightarrow d$

Frequency matrix
(21 criteria 4
alternatives)

Rank	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	Points
1st	8	7	6	0	3
2nd	0	9	5	7	2
3rd	0	5	10	6	1
4th	13	0	0	8	0

The Borda count was developed independently several times, (e.g. by Nicolaus Cusanus beginning XV century) but is named for Jean-Charles de Borda, who devised the system in 1770.

It is currently used for the election of two ethnic minority members of the National Assembly of Slovenia

(<https://www.electoral-reform.org.uk/how-do-elections-work-in-slovenia/>)

It is used throughout the world by various organisations and competitions [e.g. in academia]



Jean-Charles,
chevalier de
Borda

Borda was a mariner and a scientist. Worked on chronometers. Between 1777 and 1778, he participated in the American Revolutionary War.

The French Academy of Sciences used Borda's method to elect its members for about two decades [till Napoleon Bonaparte became president...]



Condorcet's outsourcing matrix (21 criteria 4 alternatives)

# of indicators	3	5	7	6
1st position	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>
2nd position	<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>
3rd position	<i>c</i>	<i>b</i>	<i>c</i>	<i>d</i>
4th position	<i>d</i>	<i>d</i>	<i>a</i>	<i>a</i>

Frequency matrix

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>a</i>	0	8	8	8
<i>b</i>	13	0	10	21
<i>c</i>	13	11	0	14
<i>d</i>	13	0	7	0

Outscoring matrix

B better
than a
 $7+6=13$
times

How to move from frequency to outscoring ?

# of indicators	3	5	7	6
1st position	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>
2nd position	<i>b</i>	<i>c</i>	<i>d</i>	<i>b</i>
3rd position	<i>c</i>	<i>b</i>	<i>c</i>	<i>d</i>
4th position	<i>d</i>	<i>d</i>	<i>a</i>	<i>a</i>

Frequency matrix

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>a</i>	0	8	8	8
<i>b</i>	13	0	10	21
<i>c</i>	13	11	0	14
<i>d</i>	13	0	7	0

Outscoring matrix

Condorcet's outsourcing matrix (21 criteria 4 alternatives)

For each pair of countries a concordance index is computed by counting how many indicators/voters are in favour of each country (e.g. 13 voters prefer b to a).

Note the “constant sum property” in the outranking matrix ($13+8=21$ number of indicators/voters)

	a	b	c	d
a	0	8	8	8
b	13	0	10	21
c	13	11	0	14
d	13	0	7	0

Outranking matrix

How to use Condorcet's outscoring matrix (21 criteria 4 alternatives)

Pairs with concordance index $> 50\%$ of the indicators/voters are considered: majority threshold = 11 (i.e. a number of voters $> 50\%$ of voters=21)

Thus aP none, $bPa=13$, $bPd=21$ (=*always*), $cPa=13$, $cPb=11$, $cPd=14$, $dPa=13$.

c is better than a, b, d so it is the winner

b is better than the remaining a, d , it is the second best

d is better than a .

→ Condorcet solution: $c \rightarrow b \rightarrow d \rightarrow a$

	a	b	c	d
a	0	8	8	8
b	13	0	10	21
c	13	11	0	14
d	13	0	7	0

Count row-wise discarding entries < 11 as there are 21 voters/criteria

Borda solution: $b \rightarrow c \rightarrow a \rightarrow d$

Condorcet solution: $c \rightarrow b \rightarrow d \rightarrow a$

Can we choose between Borda and Condorcet on some theoretical and/or practical grounds?



Homework

1. Both a dice and a coin are launched simultaneously in an experiment. We count a coin falling head as one and falling tail as a zero. If we call success the outcome seven (dice=six, coin=H), which is the chance of success in one experiment? Which is the chance of two successes in 4 experiments? Write down the space of the events first.
2. Solve the problem below, knowing that $\frac{\partial y}{\partial x} = 3x^2 + 2 - 4x - x^3$ and $\frac{\partial^2 y}{\partial x^2} = 6x - 4 - 3x^2$

13.4-1.* Consider the following problem:

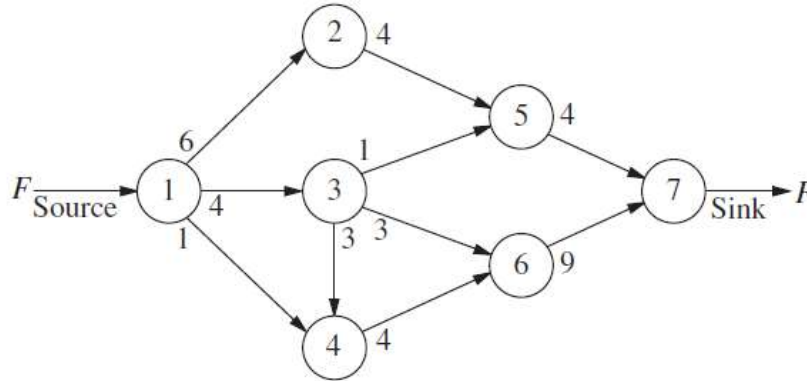
Maximize $f(x) = x^3 + 2x - 2x^2 - 0.25x^4$.

- (a) Apply the bisection method to (approximately) solve this problem. Use an error tolerance $\epsilon = 0.04$ and initial bounds $\underline{x} = 0, \bar{x} = 2.4$.
- (b) Apply Newton's method, with $\epsilon = 0.001$ and $x_1 = 1.2$, to this problem.

Choose just
one (bisection
or Newton)

Homework

3. For the network shown below, use the augmenting path algorithm described in Sec. 10.5 of Hillier to find the flow pattern giving the maximum flow from the source to the sink, given that the arc capacity from node i to node j is the number nearest node i along the arc between these nodes. Show your work.



4. Write in binary the first 10 prime numbers.

Thank you