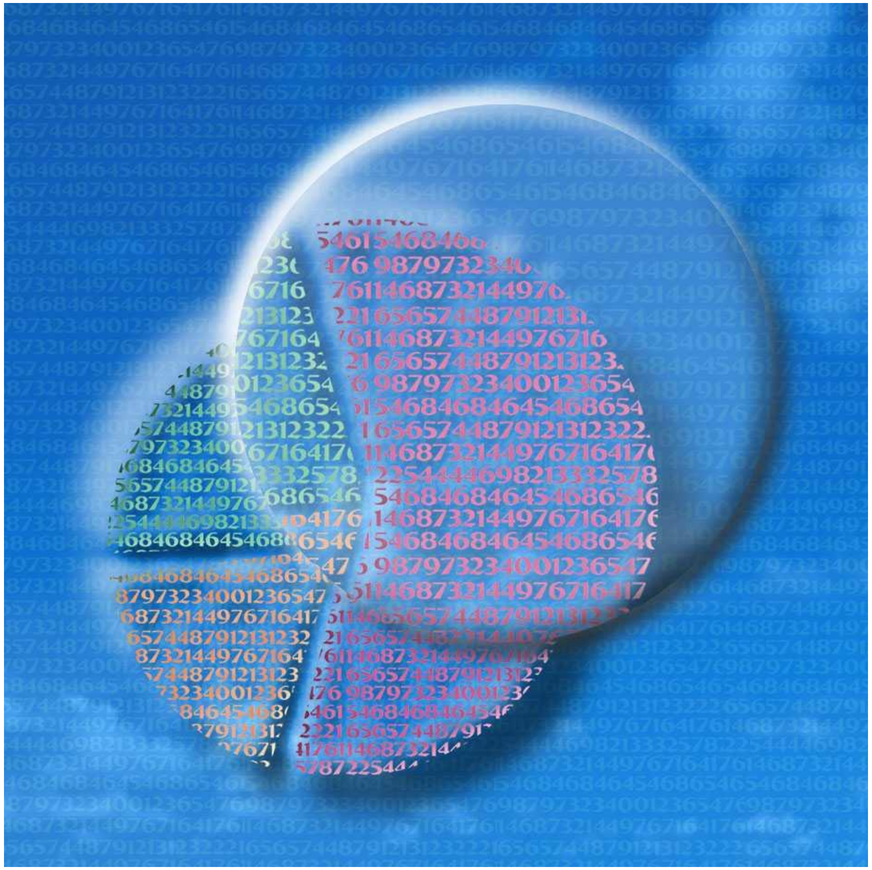


Sensitivity analysis: An introduction

Andrea Saltelli

Centre for the Study of the Sciences and the Humanities (SVT) - University of Bergen (UIB)

Institut de Ciència i Tecnologia Ambientals (ICTA) - Universitat Autonoma de Barcelona (UAB)

ROME, November 10-11 2015

CHALLENGES OF BIG DATA FOR ECONOMIC MODELING AND MANAGEMENT: TOOLS FROM EFFICIENCY ANALYSIS, SENSITIVITY ANALYSIS, SENSITIVITY AUDITING AND PHYSICS OF COMPLEX SYSTEMS

Department of Computer, Control and Management Engineering Antonio Ruberti (DIAG)

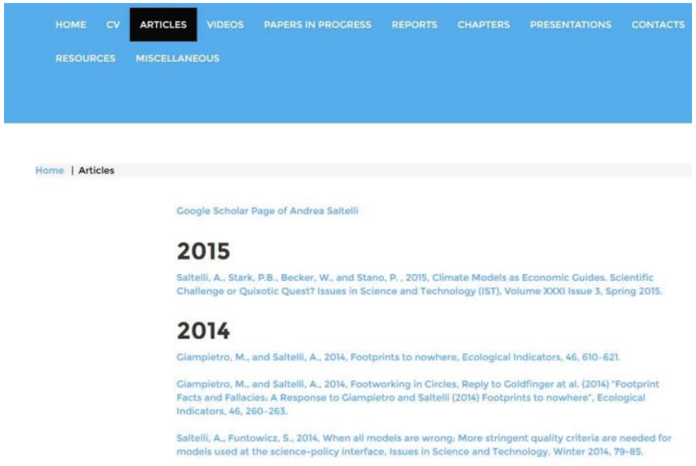
Sapienza University of Rome, Aula Magna, Via Ariosto, 25

andrea.saltelli@uib.no

andrea.saltelli@uib.no

@andreasaltelli

www.andreasaltelli.eu


HOME CV ARTICLES VIDEOS PAPERS IN PROGRESS REPORTS CHAPTERS PRESENTATIONS CONTACTS
RESOURCES MISCELLANEOUS

Home | Articles

Google Scholar Page of Andrea Saltelli

2015

Saltelli, A., Stark, P.B., Becker, W., and Stano, P. , 2015, Climate Models as Economic Guides. Scientific Challenge or Quixotic Quest? Issues in Science and Technology (IST), Volume XXXI Issue 3, Spring 2015.

2014

Giampietro, M., and Saltelli, A., 2014, Footprints to nowhere, Ecological Indicators, 46, 610-621.

Giampietro, M., and Saltelli, A., 2014, Footworking in Circles, Reply to Goldfinger at al. (2014) "Footprint Facts and Fallacies: A Response to Giampietro and Saltelli (2014) Footprints to nowhere", Ecological Indicators, 46, 260-263.

Saltelli, A., Funtowicz, S., 2014, When all models are wrong: More stringent quality criteria are needed for models used at the science-policy interface, Issues in Science and Technology, Winter 2014, 79-85.

www.andreasaltelli.eu

HOME CV ARTICLES VIDEOS PAPERS IN PROGRESS REPORTS CHAPTERS PRESENTATIONS CONTACTS CONFERENCES

RESOURCES PERSONAL

Journals

Presentations

Welcome to the home page of Andrea Saltelli

Caeteris are never paribus

Where to find materials

andrea.saltelli@uib.no

sensitivity analysis,
sensitivity auditing,
science for policy,
impact assessment, ...

Topics for this course

Sensitivity Analysis

Sensitivity Auditing

Impact Assessment

Science advice

Science’s crisis

Science and Technology Studies (STS)

Topics for this course

Sensitivity Analysis ←

Sensitivity Auditing

Impact Assessment

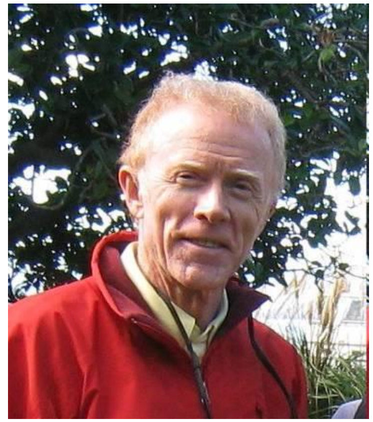
Science advice

Science's crisis

Science and Technology Studies (STS)

Narratives of Innovation

When testing the evidence behind inference some reasonable people (and guidelines) suggest that ‘sensitivity analysis would help’



...

JRC fostered sensitivity analysis development and uptake (20 years of papers, schools and books).

Today we call it **sensitivity auditing** and teach it within the syllabus for impact assessment run by the secretary general of the European Commission.

Edward E. Leamer, 1990, Let's Take the Con Out of Econometrics, American Economics Review, 73 (March 1983), 31-43.

<<I have proposed a form of organised sensitivity analysis that I call “global sensitivity analysis” in which a neighborhood of alternative assumptions is selected and the corresponding interval of inferences is identified.>>

Edward E. Leamer, 1990, Let's Take the Con Out of Econometrics, American Economics Review, 73 (March 1983), 31-43.

<<Conclusions are judged to be sturdy only if the neighborhood of assumptions is wide enough to be credible and the corresponding interval of inferences is narrow enough to be useful.>>

Funtowicz & Ravetz's GIGO (Garbage In, Garbage Out) Science – or pseudo-science – "where uncertainties in inputs must be suppressed least outputs become indeterminate"

Leamer's 'Conclusions are judged to be sturdy only if the neighborhood of assumptions is wide enough to be credible and the corresponding interval of inferences is narrow enough to be useful'.

With the ashes of the mathematical models used to rate mortgage-backed securities still smoldering on Wall Street, now is an ideal time to revisit the sensitivity issues.

Tantalus on the Road to Asymptopia
Edward E. Leamer, 2010 *Journal of Economic Perspectives*, **24**, (2), 31–46.

"… my observation of economists at work who routinely pass their data through the filters of many models and then choose a few results for reporting purposes."

Ibidem

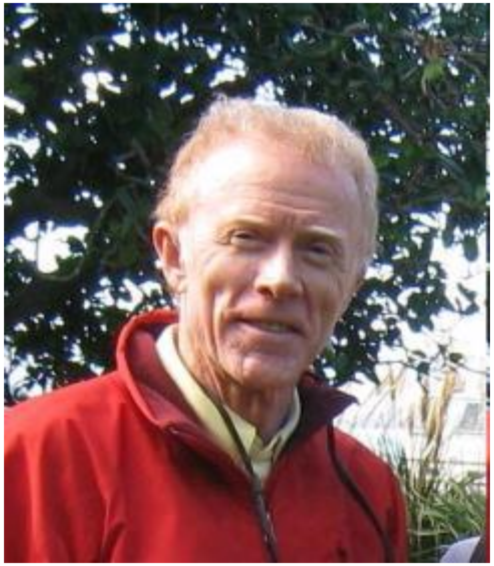
“One reason these methods are rarely used is their honesty seems destructive;”

Ibidem

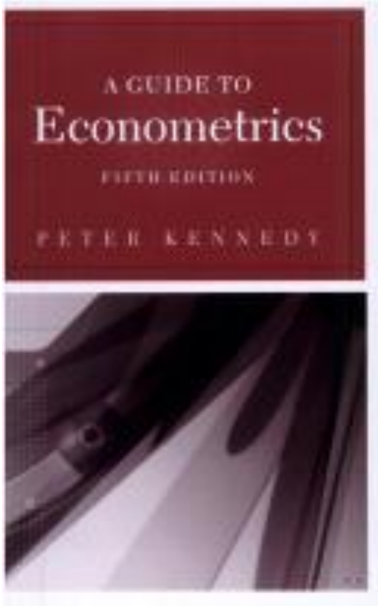
“or, to put it another way, a fanatical commitment to fanciful formal models is often needed to create the appearance of progress.”

Ibidem

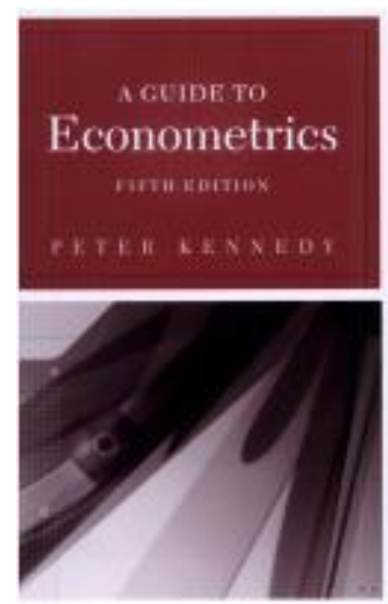
Peter Kennedy, A Guide to Econometrics.
Anticipating criticism by applying sensitivity analysis. This is one of the ten commandments of applied econometrics according to Peter Kennedy:

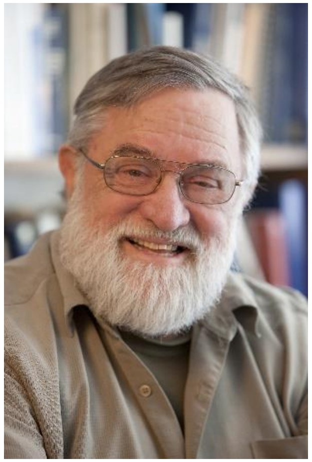


<<Thou shall confess in the presence of sensitivity.
Corollary: Thou shall anticipate criticism >>



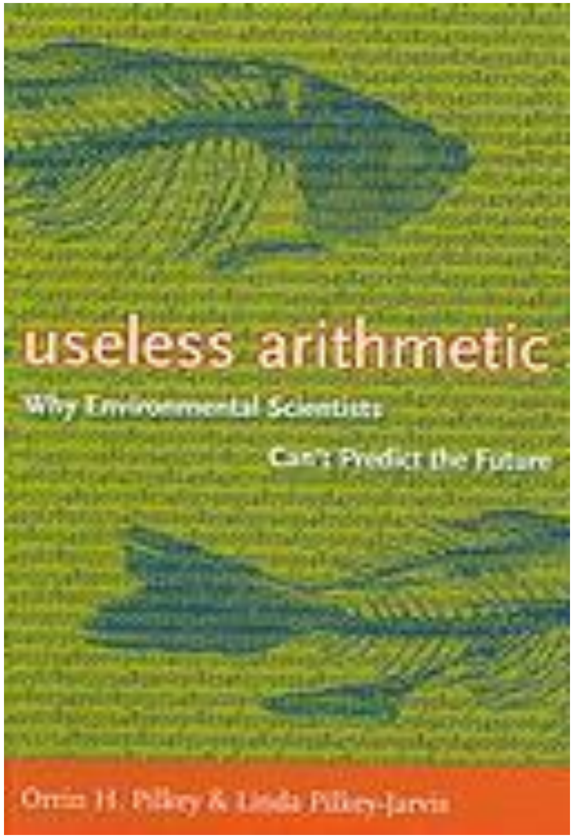
<<When reporting a sensitivity analysis, researchers should explain fully their specification search so that the readers can judge for themselves how the results may have been affected. This is basically an `honesty is the best policy' approach, […]’.>>



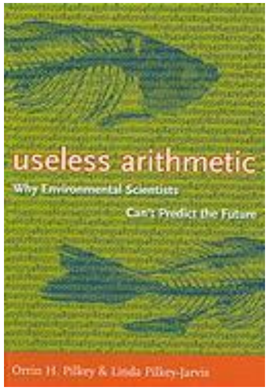


Orrin H. Pilkey
Duke University, NC

Useless Arithmetic: Why Environmental Scientists Can't Predict the Future
by Orrin H. Pilkey and Linda Pilkey-Jarvis



‘Quantitative mathematical models used by policy makers and government administrators to form environmental policies are seriously flawed’

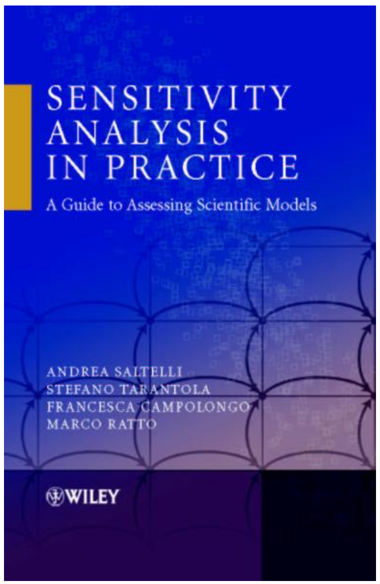


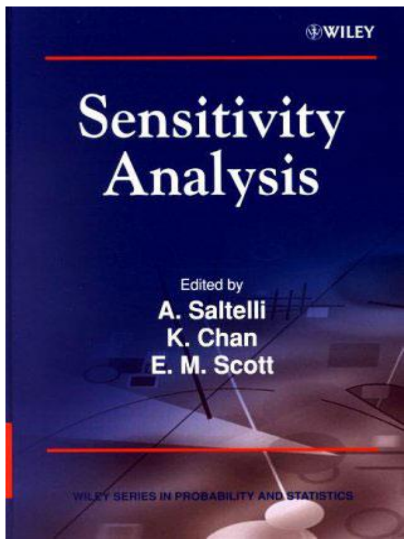
<<It is important, however, to recognize that the sensitivity of the parameter in the equation is what is being determined, not the sensitivity of the parameter in nature.

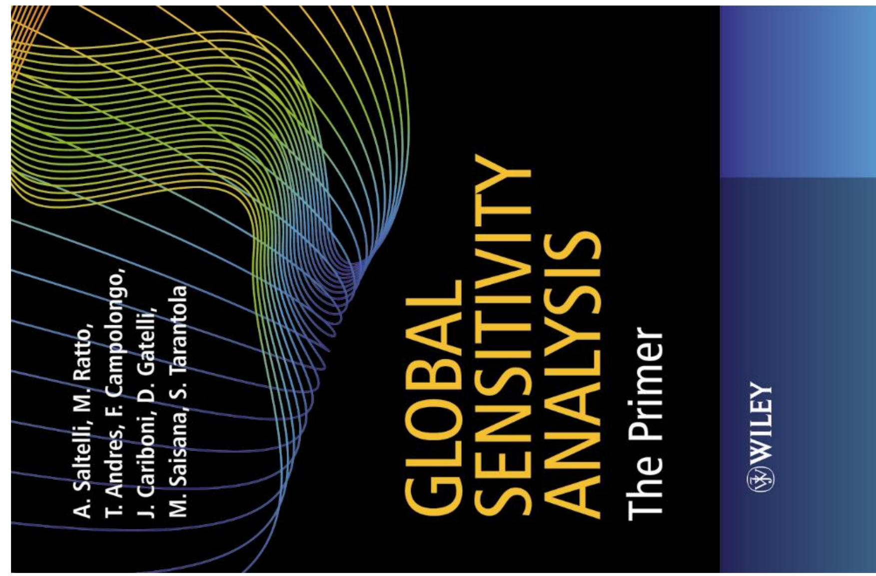
[…] If the model is wrong or if it is a poor representation of reality, determining the sensitivity of an individual parameter in the model is a meaningless pursuit.>>

JRC fostered sensitivity analysis development and uptake (20 years of papers, schools and books).

Today we expand its concept to **sensitivity auditing** and teach it within the syllabus for impact assessment run by the secretary general of the European Commission.







With a network of practitioners we organize international conferences on sensitivity analysis every 3 years since 1995, next November 2016

Summer schools every two years – next July 2016

From: Saltelli, A., Annoni P., 2010
How to avoid a perfunctory sensitivity analysis,
Environmental Modeling and Software, **25**, 1508-1517.

What do these have in common?

J. Campbell, *et al.*, *Science* **322**, 1085 (2008).
R. Bailis, M. Ezzati, D. Kammen, *Science* **308**, 98 (2005).
E. Stites, P. Trampont, Z. Ma, K. Ravichandran, *Science* **318**, 463 (2007).
J. Murphy, et al., *Nature* **430**, 768-772 (2004).
J. Coggan, *et al.*, *Science* **309**, 446 (2005).

OAT

OAT methods – derivatives – local

$$S_1 = \frac{\partial y}{\partial x_j}$$

Effect on Y of perturbing x_j around its nominal value x_j^0

$$S_2 = \frac{\partial y}{\partial x_j}\frac{x_j^0}{y^0}$$

Relative effect on Y of perturbing x_j by a fixed fraction of its nominal value x_j^0

$$S_3 = \frac{\partial y}{\partial x_j}\frac{std(x_j)}{std(y)}$$

Relative effect on Y of perturbing x_j by a fixed fraction of its standard deviation

Why not just changing one factor at a time (OAT)?

•**OMB A4**: Use a numerical sensitivity analysis to examine how the results of your analysis vary with plausible changes in assumptions, choices of input data, and alternative analytical approaches. Sensitivity analysis is especially valuable when the information is lacking to carry out a formal probabilistic simulation. Sensitivity analysis can be used to find 'switch points' -- critical parameter values at which estimated net benefits change sign or the low cost alternative switches. Sensitivity analysis usually proceeds by changing one variable or assumption at a time, but it can also be done by varying a combination of variables simultaneously to learn more about the robustness of your results to widespread changes. Again, however, major rules above the $1 billion annual threshold require a formal treatment.

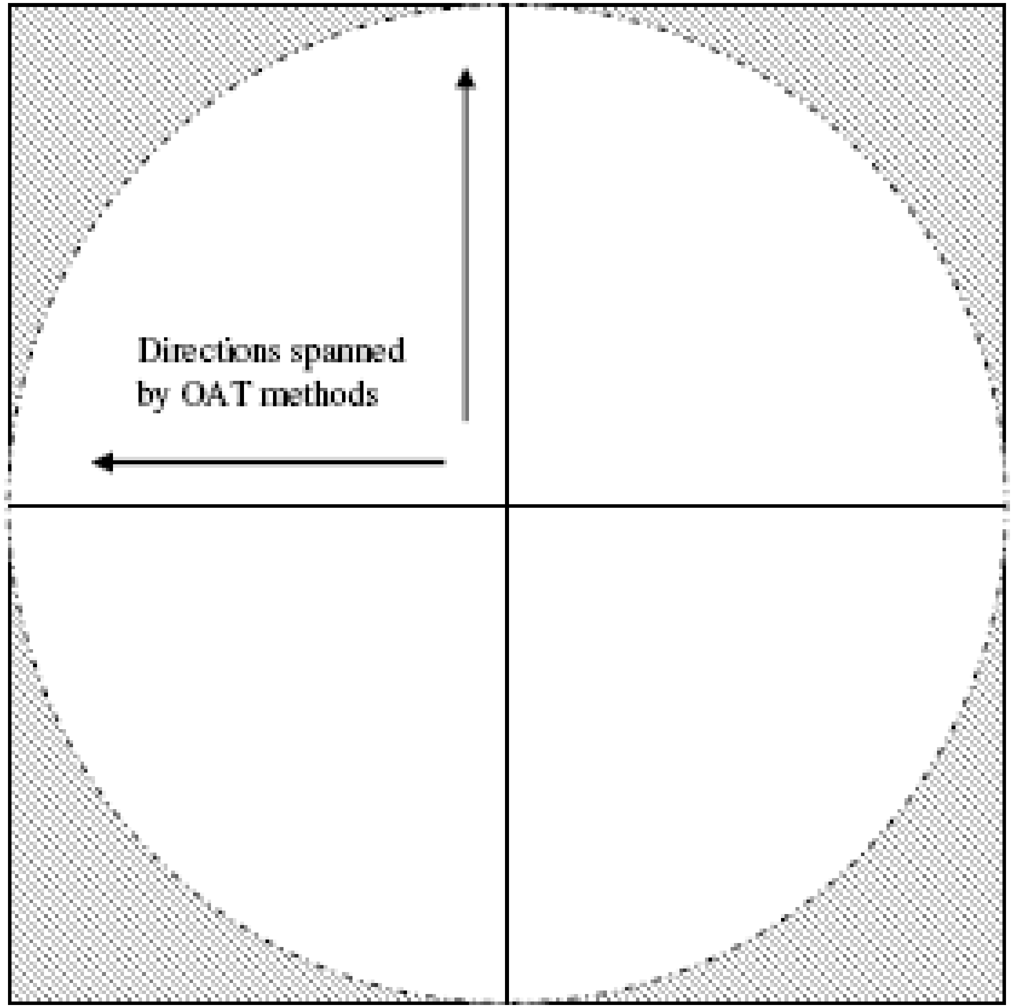
http://www.whitehouse.gov/omb/circulars_a004_a-4/ 2003

Why not just changing one factor at a time (OAT)?

- **OMB A4**: […]Sensitivity analysis usually proceeds by changing one variable or assumption at a time, but it can also be done by varying a combination of variables simultaneously to learn more about the robustness of your results to widespread changes.

http://www.whitehouse.gov/omb/circulars_a004_a-4/ 2003

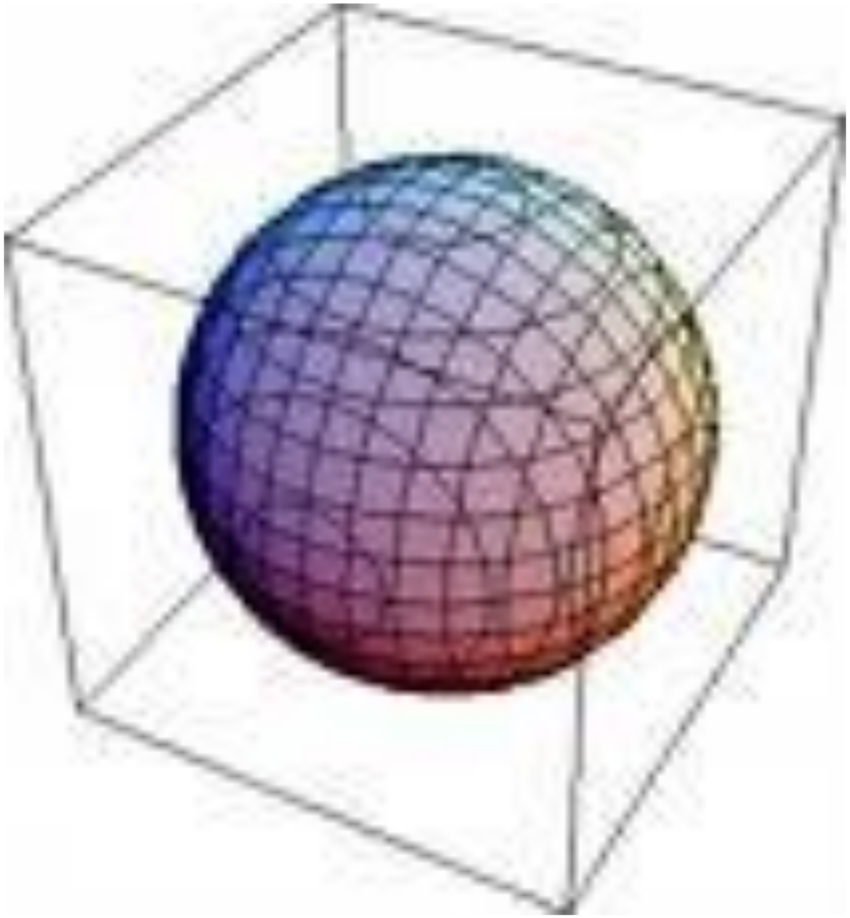
OAT in 2 dimensions



Area circle / area square =?

~ 3/4

OAT in 3 dimensions



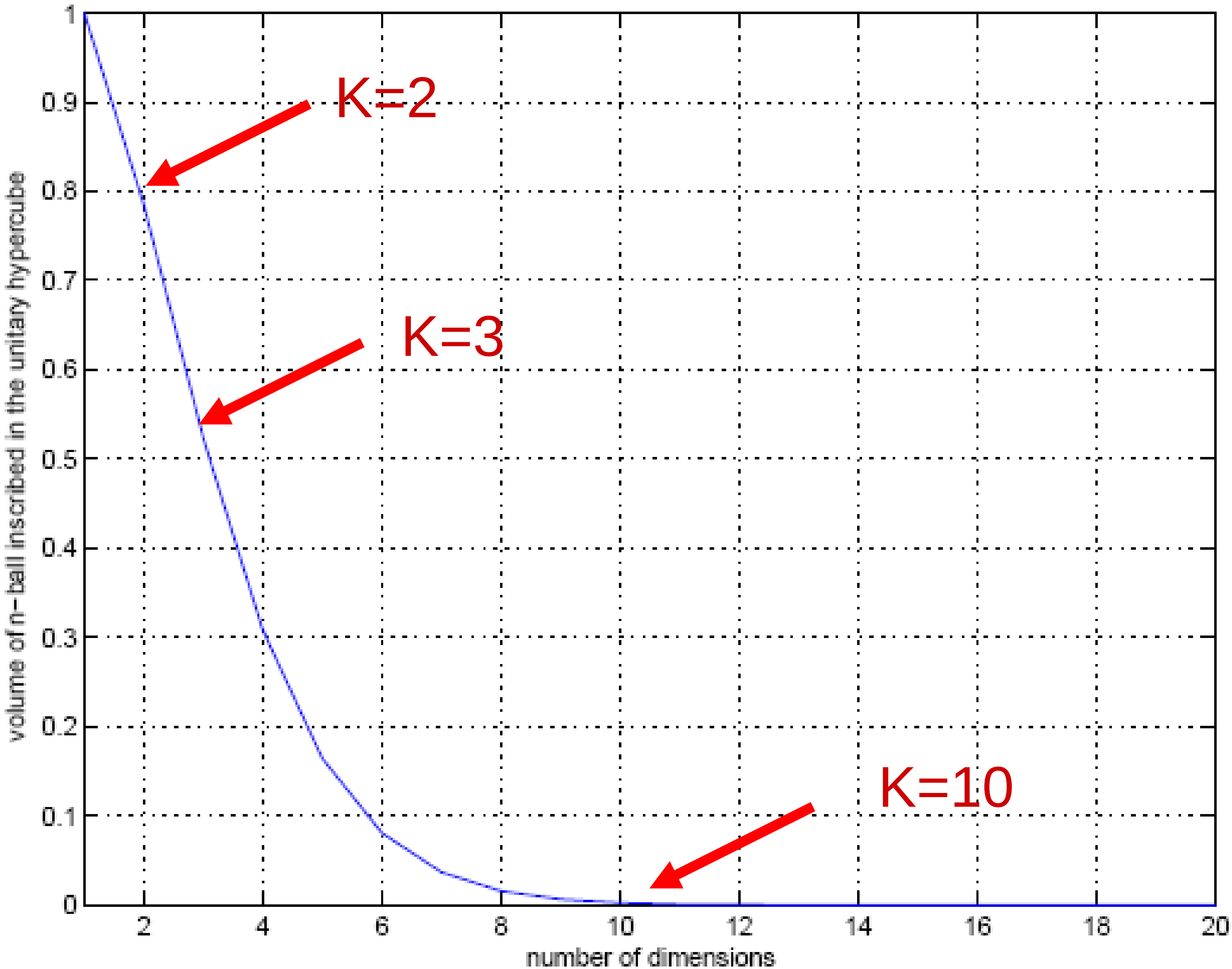
Volume sphere / volume cube =?

~ 1/2

OAT in 10 dimensions

Volume hypersphere / volume ten dimensional hypercube ~ 0.0025

OAT in k dimensions



An environmental case study

Describe a chain of species mutating one into another without backward reactions

species 1 $\rightarrow$ species 2 … species k

An environmental case study

The Bateman equations describe the concentrations N_i of k species in linear chain governed by rate constants λ_i:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\lambda_1 N_1$$

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = \lambda_{i-1} N_{i-1} - \lambda_i N_i \qquad (i = 2, \ldots k)$$

$$\text{if } N_1 \neq 0 \text{ and } N_i = 0 \; \forall \, i > 1$$

We want to simulate $N_k(t)$ with different k

$$N_k(t) = \frac{N_1(0)}{\lambda_k} \sum_{i=1}^{k} \lambda_i \alpha_i \exp(-\lambda_i t)$$

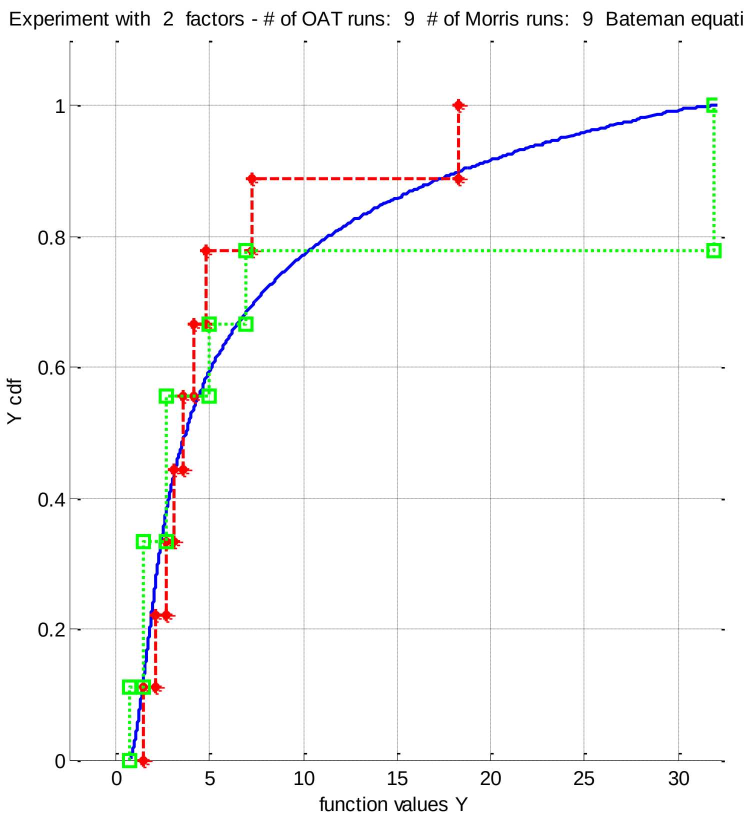
$$\alpha_i = \prod_{j=i \; j \neq i}^{k} \frac{\lambda_j}{(\lambda_j - \lambda_i)}$$

An environmental case study

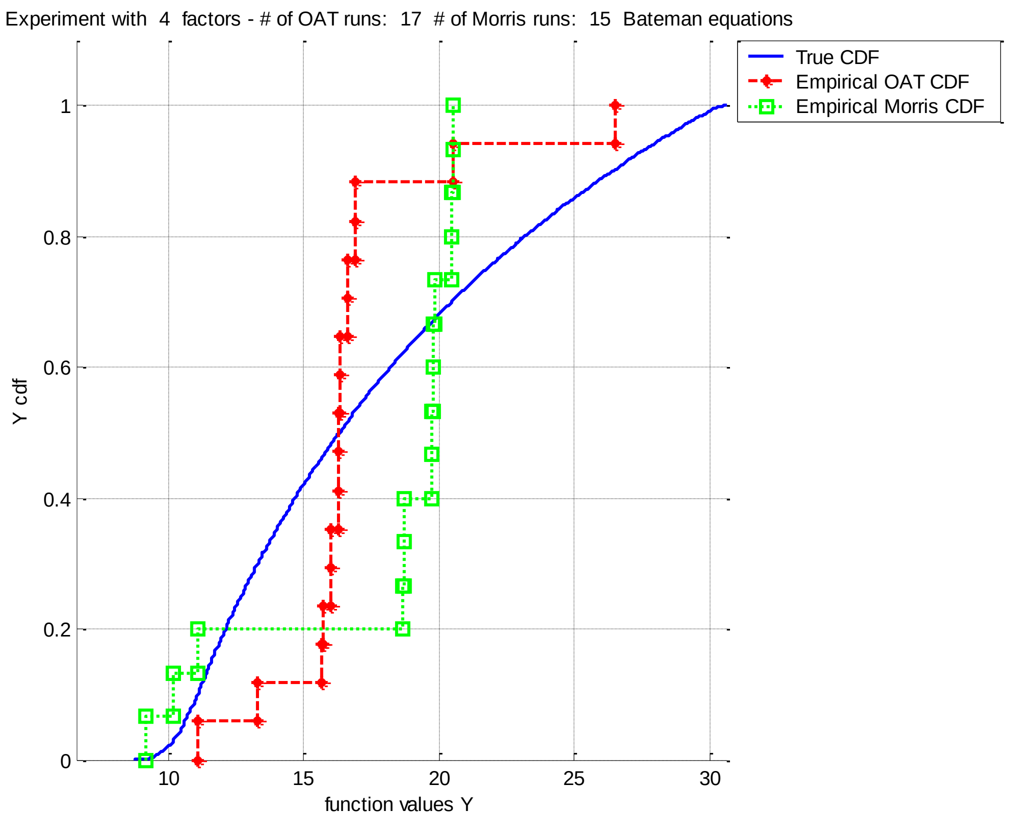
Our settings:

- Six experiments with increasing number of species k involved
- Fixed time instant t
- λ_i randomly sampled from an uniform U[1,100]
- Concentration of initial species $N_1(0)$=100
- Comparison between OAT and a global method with (roughly) the same number of runs

2 species

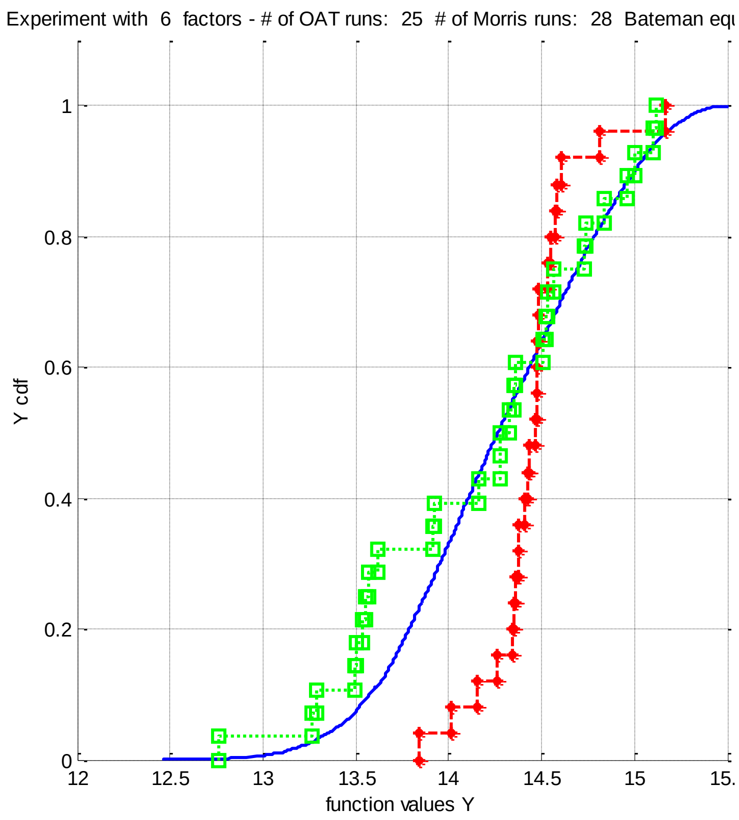


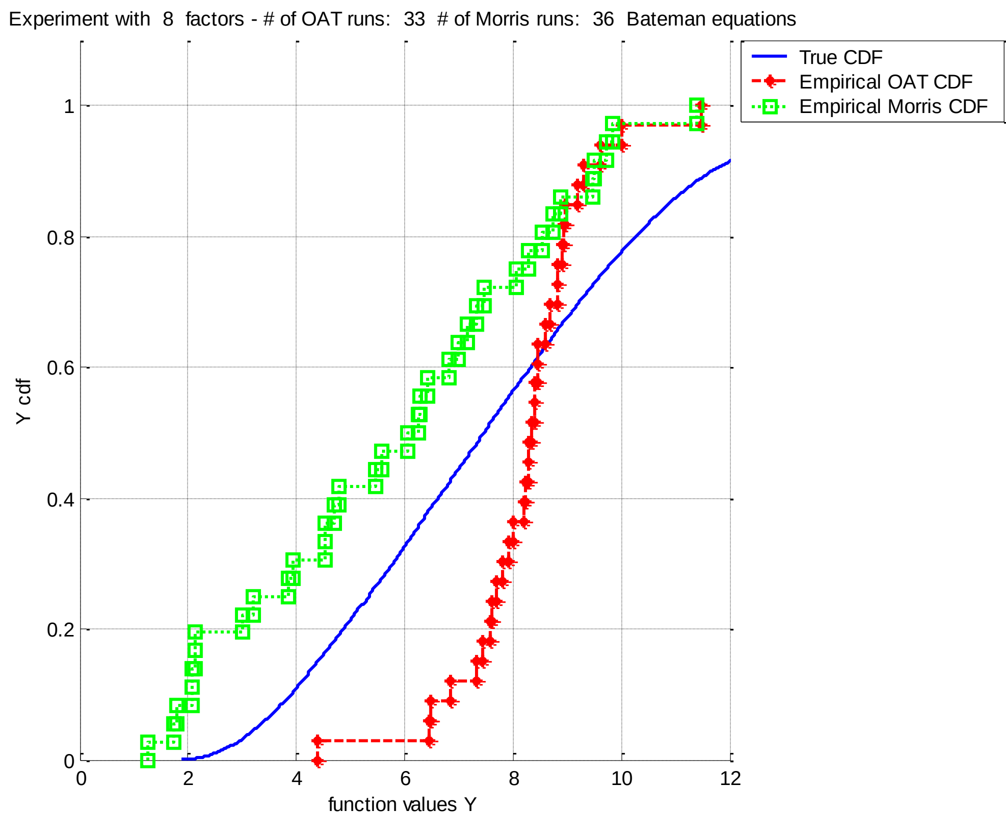
4 species



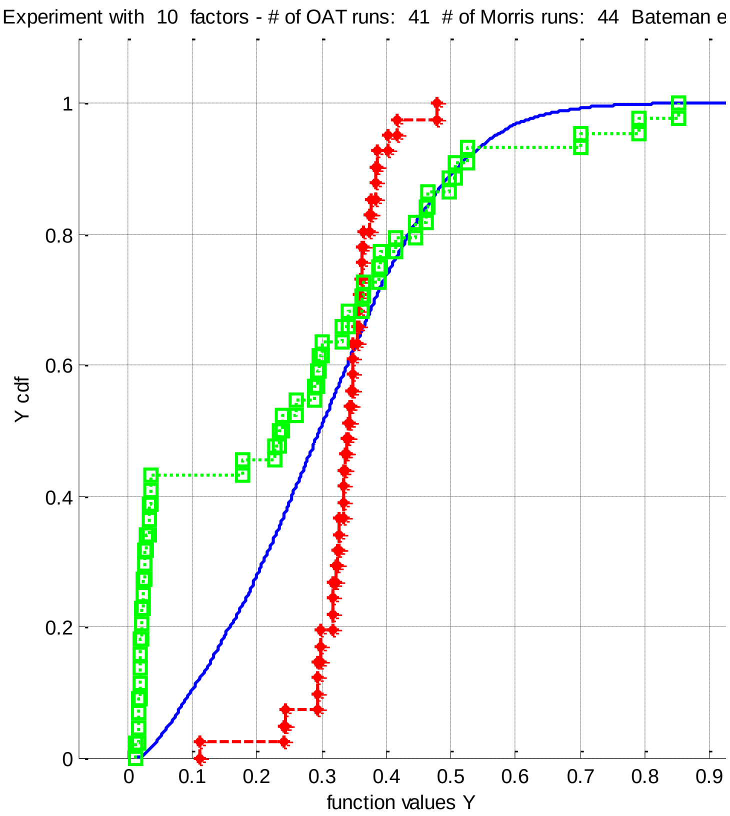
6 species

8 species

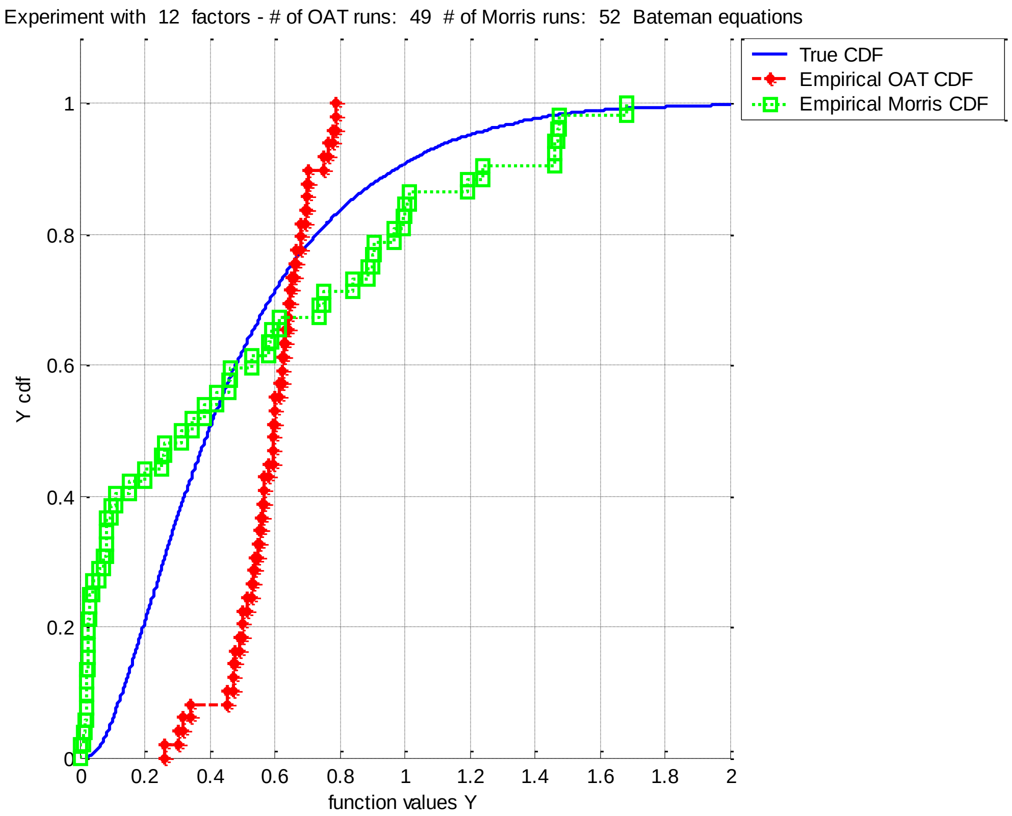




10 species



12 species



How are we doing in 2015?

…OAT is still the most largely used technique in SA, … clear increase in the use of GSA with preference for regression and variance-based techniques … even after adjusting for the growth of publications in the sole modeling field.

Ferretti, F., Saltelli A., Tarantola, S., 2015, Trends in Sensitivity Analysis practice in the last decade, submitted to: Science of the Total Environment, special issue on Human and biota exposure.

Definition of uncertainty and sensitivity analysis.

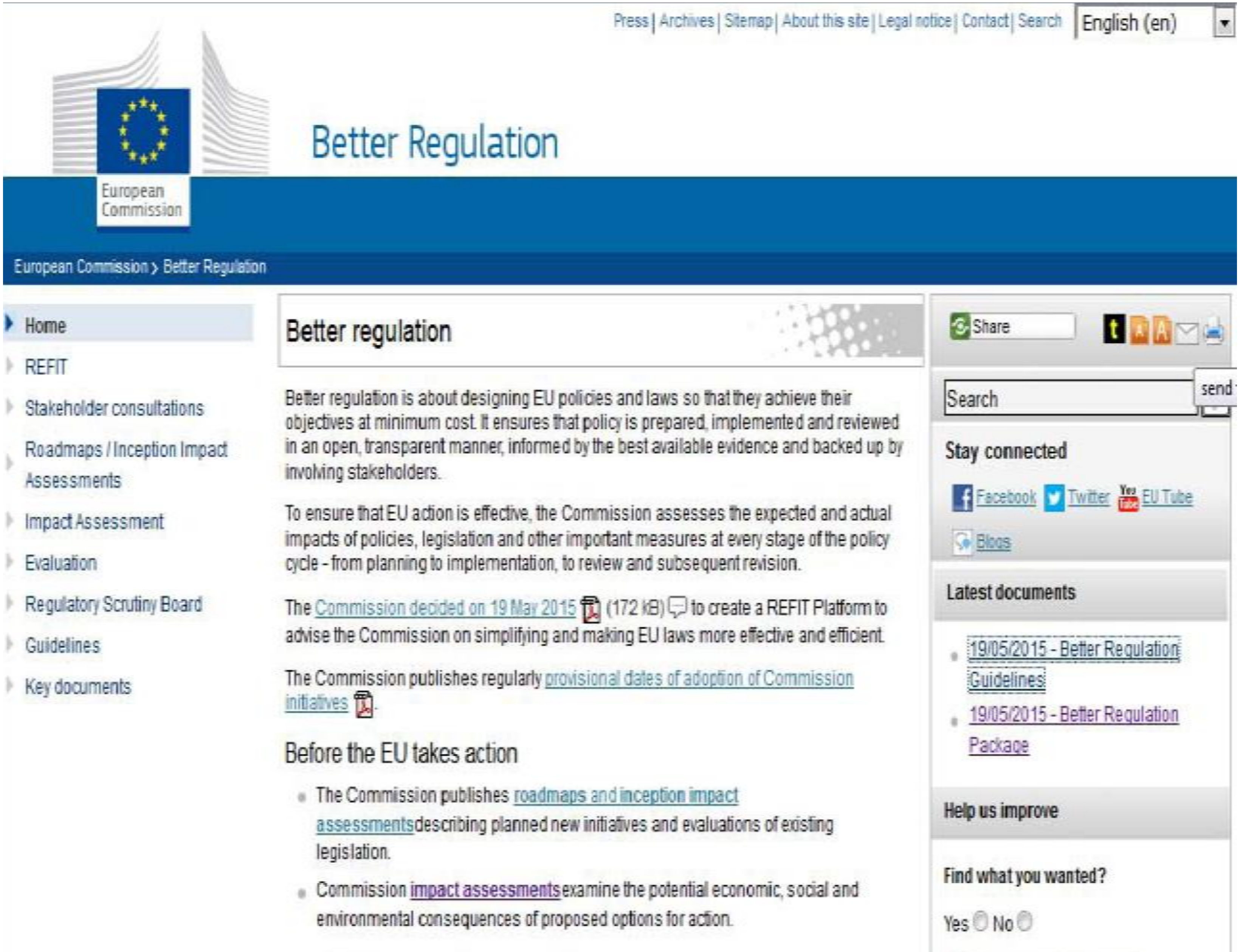
Sensitivity analysis: The study of the relative importance of different input factors on the model output.

Uncertainty analysis: Focuses on just quantifying the uncertainty in model output.

[Global*] sensitivity analysis: “The study of how the uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input”

Saltelli A., 2002, Sensitivity Analysis for Importance Assessment, Risk Analysis, 22 (3), 1-12.

http://ec.europa.eu/smart-regulation/


Press | Archives | Sitemap | About this site | Legal notice | Contact | Search English (en)

European Commission

Better Regulation

European Commission > Better Regulation

- Home
- REFIT
- Stakeholder consultations
- Roadmaps / Inception Impact Assessments
- Impact Assessment
- Evaluation
- Regulatory Scrutiny Board
- Guidelines
- Key documents

Better regulation

Better regulation is about designing EU policies and laws so that they achieve their objectives at minimum cost. It ensures that policy is prepared, implemented and reviewed in an open, transparent manner, informed by the best available evidence and backed up by involving stakeholders.

To ensure that EU action is effective, the Commission assesses the expected and actual impacts of policies, legislation and other important measures at every stage of the policy cycle - from planning to implementation, to review and subsequent revision.

The Commission decided on 19 May 2015 (172 kB) to create a REFIT Platform to advise the Commission on simplifying and making EU laws more effective and efficient.

The Commission publishes regularly provisional dates of adoption of Commission initiatives.

Before the EU takes action

- The Commission publishes roadmaps and inception impact assessments describing planned new initiatives and evaluations of existing legislation.
- Commission impact assessments examine the potential economic, social and environmental consequences of proposed options for action.

Share

Search

send

Stay connected

Facebook Twitter EU Tube

Blogs

Latest documents

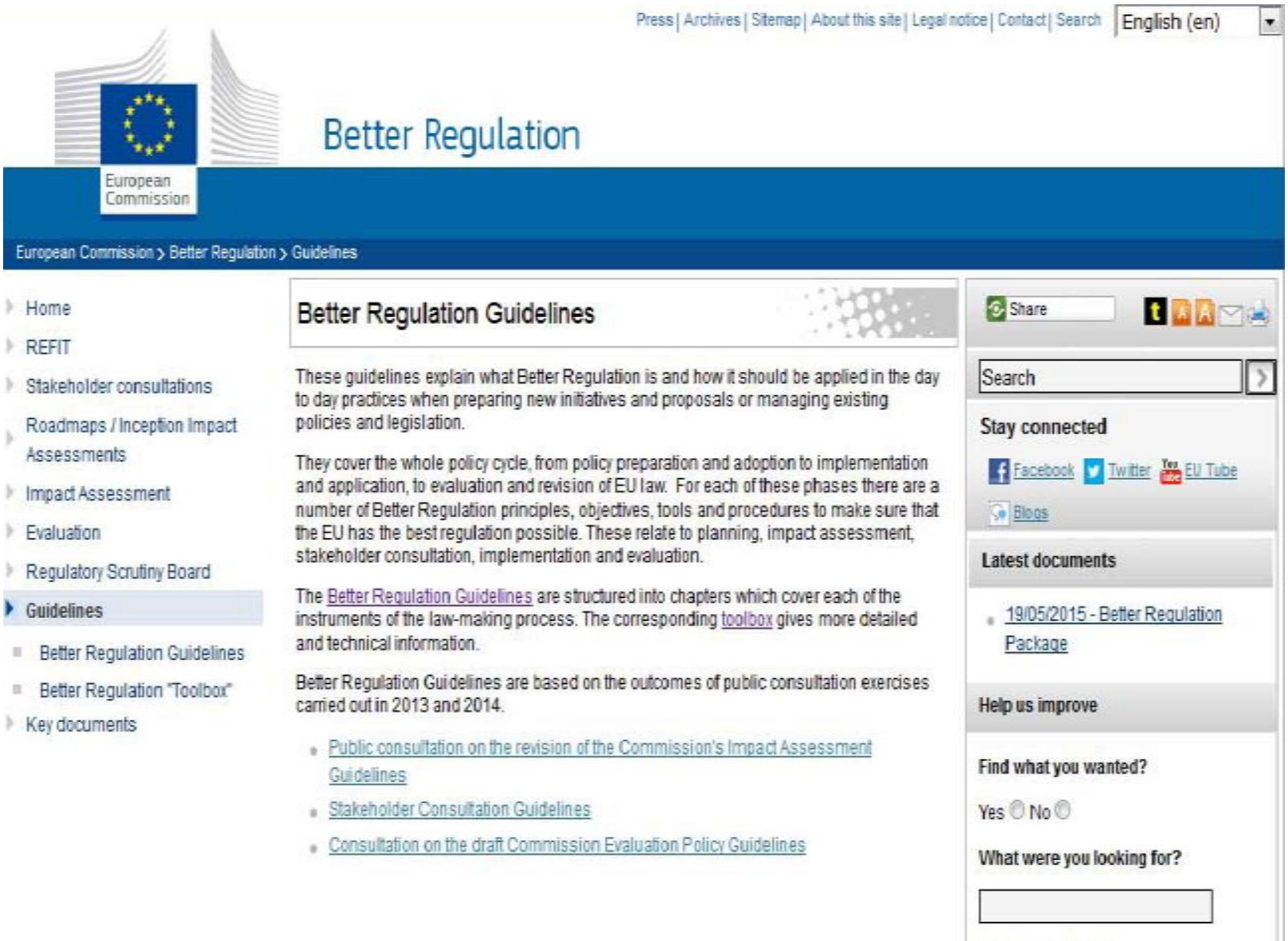
- 19/05/2015 - Better Regulation Guidelines
- 19/05/2015 - Better Regulation Package

Help us improve

Find what you wanted?

Yes No

http://ec.europa.eu/smart-regulation/guidelines/docs/br_toolbox_en.pdf


Press | Archives | Sitemap | About this site | Legal notice | Contact | Search   English (en)

European Commission

Better Regulation

European Commission > Better Regulation > Guidelines

- Home
- REFIT
- Stakeholder consultations
- Roadmaps / Inception Impact Assessments
- Impact Assessment
- Evaluation
- Regulatory Scrutiny Board
- **Guidelines**
  - Better Regulation Guidelines
  - Better Regulation "Toolbox"
- Key documents

Better Regulation Guidelines

These guidelines explain what Better Regulation is and how it should be applied in the day to day practices when preparing new initiatives and proposals or managing existing policies and legislation.

They cover the whole policy cycle, from policy preparation and adoption to implementation and application, to evaluation and revision of EU law. For each of these phases there are a number of Better Regulation principles, objectives, tools and procedures to make sure that the EU has the best regulation possible. These relate to planning, impact assessment, stakeholder consultation, implementation and evaluation.

The Better Regulation Guidelines are structured into chapters which cover each of the instruments of the law-making process. The corresponding toolbox gives more detailed and technical information.

Better Regulation Guidelines are based on the outcomes of public consultation exercises carried out in 2013 and 2014.

- Public consultation on the revision of the Commission's Impact Assessment Guidelines
- Stakeholder Consultation Guidelines
- Consultation on the draft Commission Evaluation Policy Guidelines

Share

Search

Stay connected

Facebook Twitter EU Tube

Blogs

Latest documents

- 19/05/2015 - Better Regulation Package

Help us improve

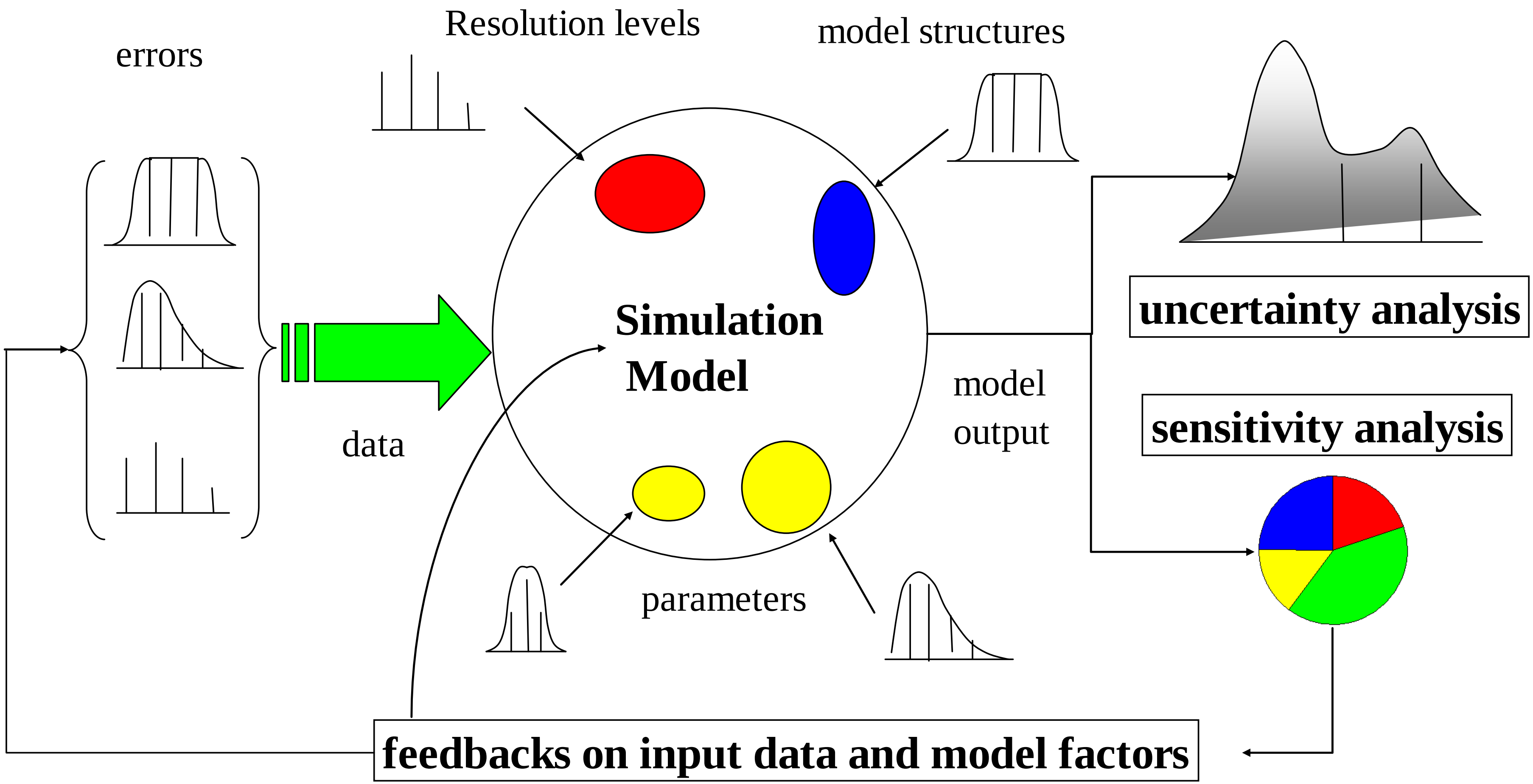
Find what you wanted?

Yes No

What were you looking for?

- Modelling in a Monte Carlo framework using quasi MC-points
- All uncertainties activated simultaneously; uncertainty and sensitivity together

An engineer's vision of UA, SA

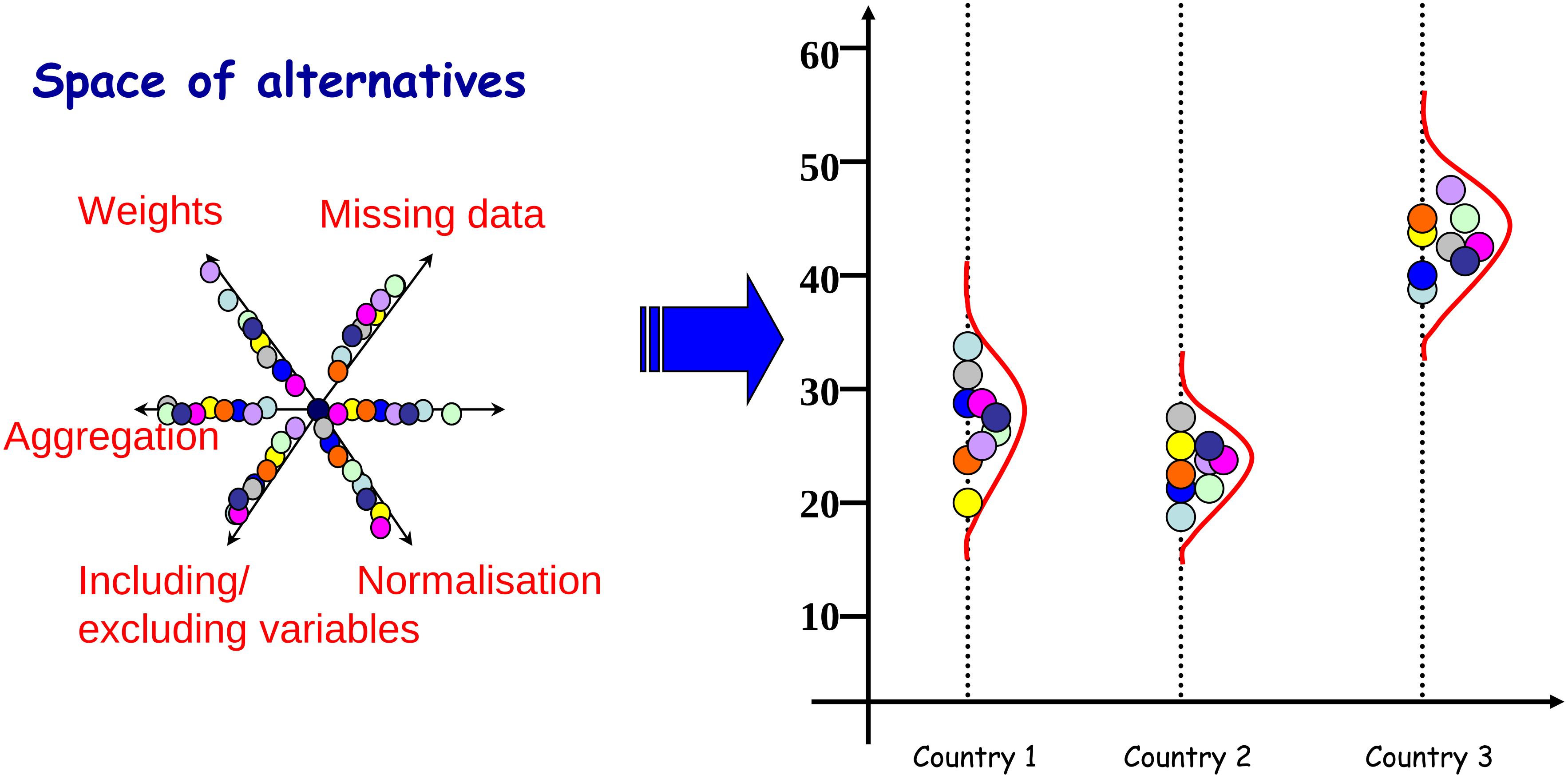


One can sample more than just factors ···

Using triggers one can sample modelling assumptions ···

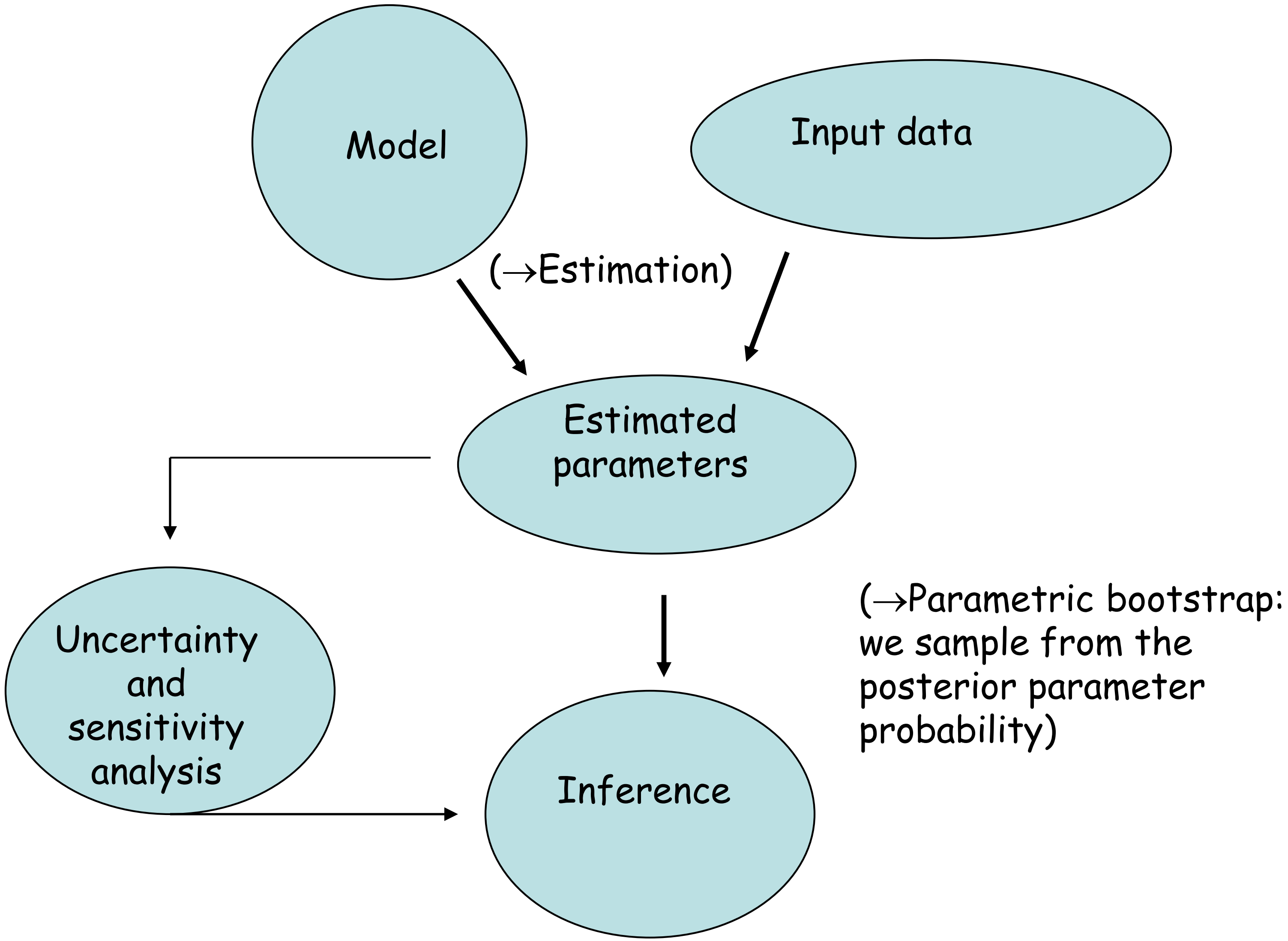
Example: Y is a composite indicator

Assumption	Alternatives
Number of indicators	▪ all six indicators included or one-at-time excluded (6 options)
Weighting method	▪ original set of weights, ▪ factor analysis, ▪ equal weighting, ▪ data envelopment analysis
Aggregation rule	▪ additive, ▪ multiplicative, ▪ Borda multi-criterion



Space of alternatives
Weights
Missing data
Aggregation
Including/
excluding variables
Normalisation
60
50
40
30
20
10
Country 1
Country 2
Country 3

Models maps assumptions onto inferences … (Parametric bootstrap version of UA/SA)



Sample matrix for parametric bootstrap.

$$\begin{array}{ccc} x_{11} & x_{12} \cdots & x_{1k} \\ x_{21} & x_{21} \cdots & x_{21} \\ \cdots & \cdots & \cdots \\ x_{N1} & x_{N1} & x_{N1} \end{array}$$

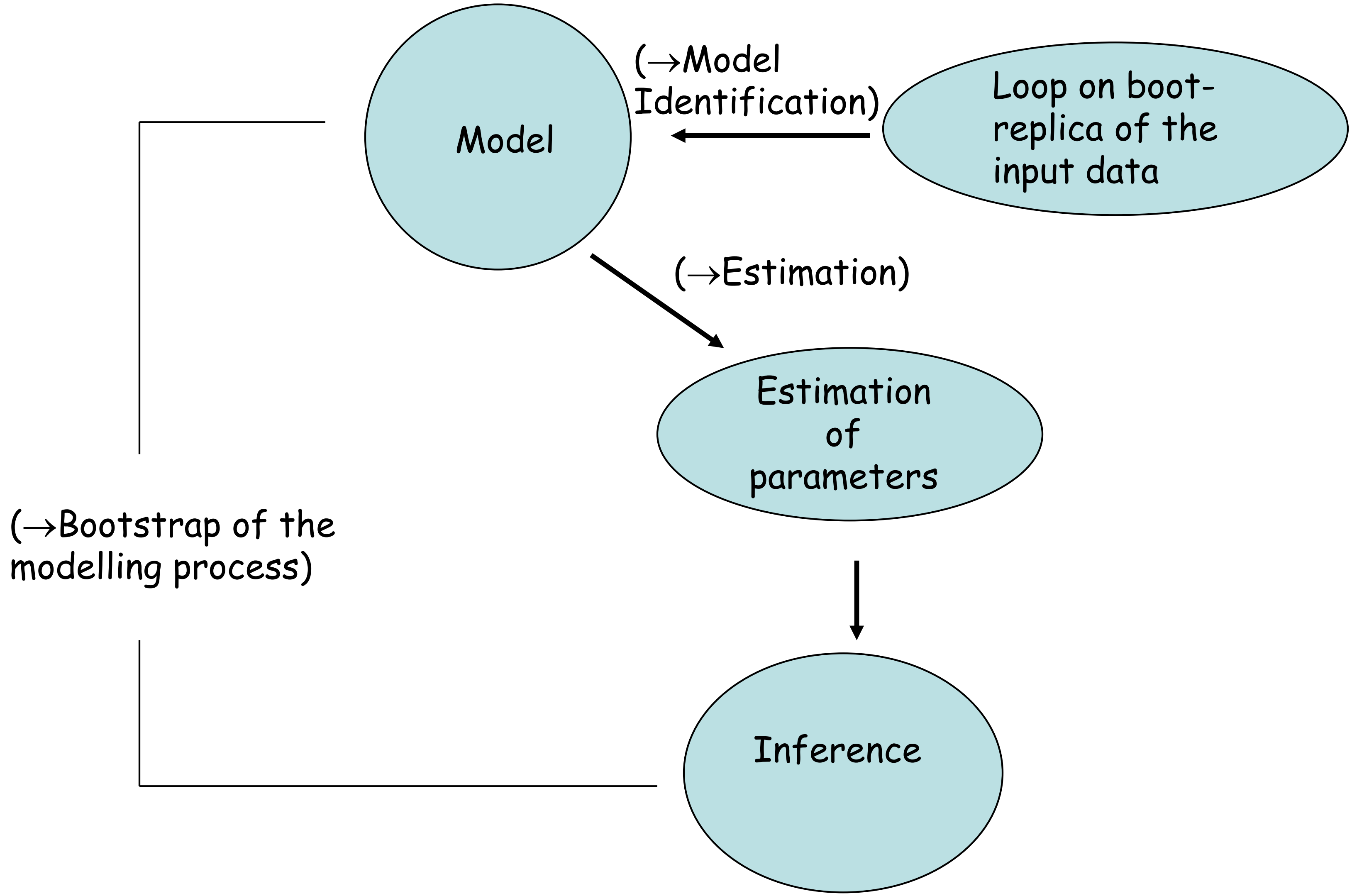
Each row is a sample trial for one model run. Each column is a sample of size N from the marginal distribution of the parameters as generated by the estimation procedure.

Model results:

Each row is the error-free result of the model run.

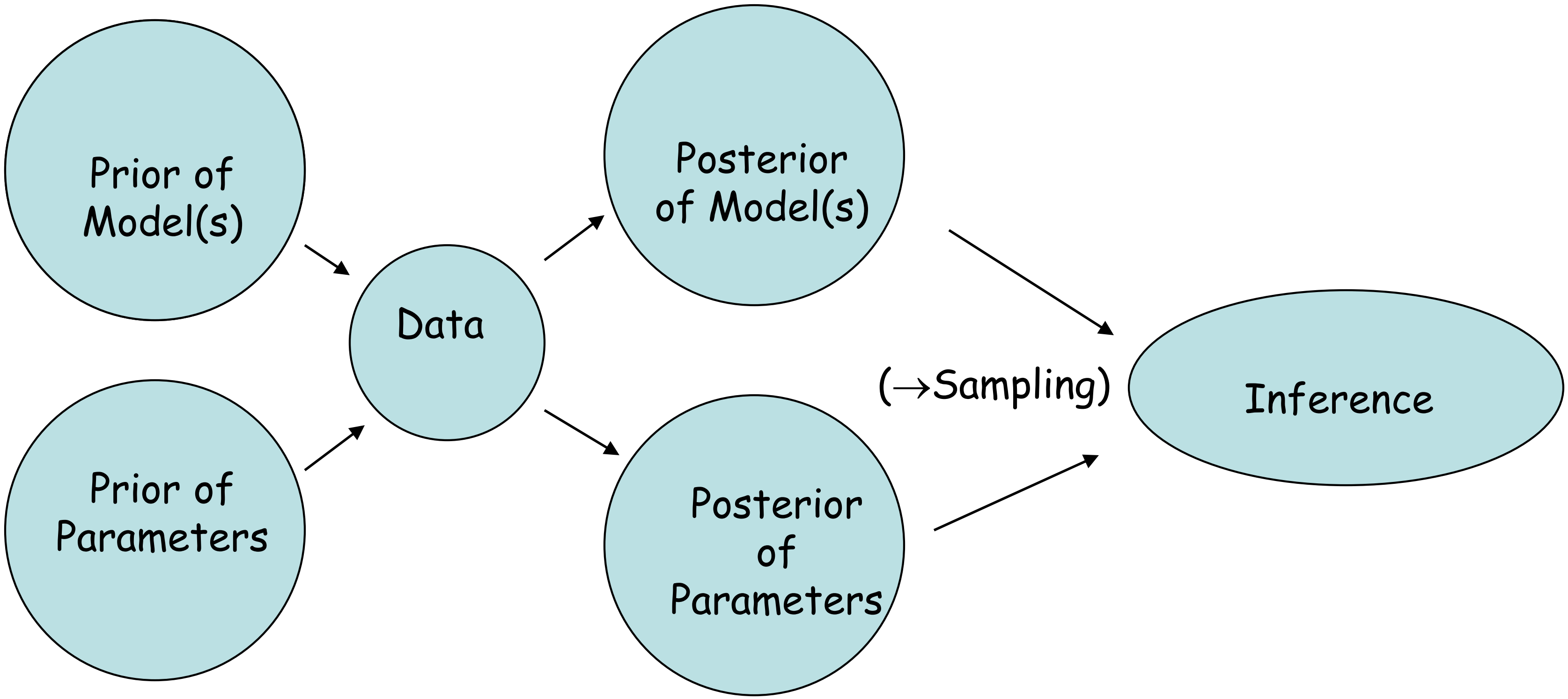
$$y_1$$
$$y_2$$
$$\cdots$$
$$y_N$$

Bootstrapping-of-the-modelling-process

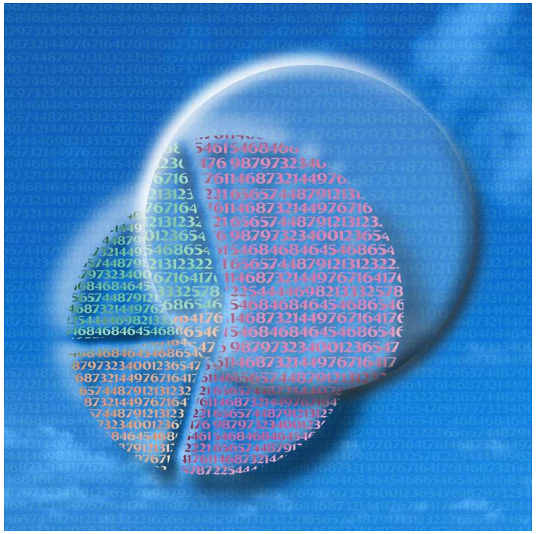


Chatfield, C., 1995, Model Uncertainty, Data Mining and Statistical Inference, Journal of the Royal Statistical Society. Series A (Statistics in Society), 158, No. 3, 419-466.

Bayesian Model Averaging



Hoeting, J.A., Madigan, D., Raftery, A.E. and Volinsky, C.T., 1999, Bayesian Model Averaging: A Tutorial
Statistical Science, 1999, Vol. 14, No. 4, 382–417



Our preferred methods for SA: variance based

Variance based methods' best formalization is based on the work of Ilya M. Sobol' (1990), who extended the work of R.I. Cukier (1973).

Total sensitivity indices by T. Homma and myself (1996).

Today a rich literature and many investigators on the topic.

Theory:

Sobol', I. M. (1990) Sensitivity estimates for nonlinear mathematical models. Matematicheskoe Modelirovanie 2, 112–118 (in Russian). [Transl. (1993) Sensitivity analysis for non-linear mathematical models, Mathematical Modelling and Computational Experiments, 1, 407–414.] Available at: http://www.andreasaltelli.eu/file/repository/sobol1993.pdf

Homma T., and Saltelli, A. (1996), "Importance Measures in Global Sensitivity Analysis of Model Output," Reliability Engineering and System Safety, 52, 1–17.

Saltelli, A., Tarantola, S., and Campolongo, F. (2000), "Sensitivity Analysis as an Ingredient of Modelling," Statistical Science, 15, 1–20.

Saltelli, A., M. Ratto, S. Tarantola and F. Campolongo, 2012 (Perennial Review of the 2005 paper), Sensitivity Analysis for Chemical Models, Chemical Reviews, 112 (5), pp PR1–PR21.

Algorithmic implementation:

Saltelli, A. (2002), "Making Best Use of Model Evaluations to Compute Sensitivity Indices," Computer Physics Communications, 145, 280–297.

A. Saltelli, P. Annoni, I. Azzini, F. Campolongo, M. Ratto, S. Tarantola, Variance based sensitivity analysis of model output. Design and estimator for the total sensitivity index, Computer Physics Communication 181 (2) (2010) 259–270.

An intuitive derivation of sensitivity indices

x_{11} x_{12} ... x_{1k}

x_{21} x_{21} ... x_{21}

... ...

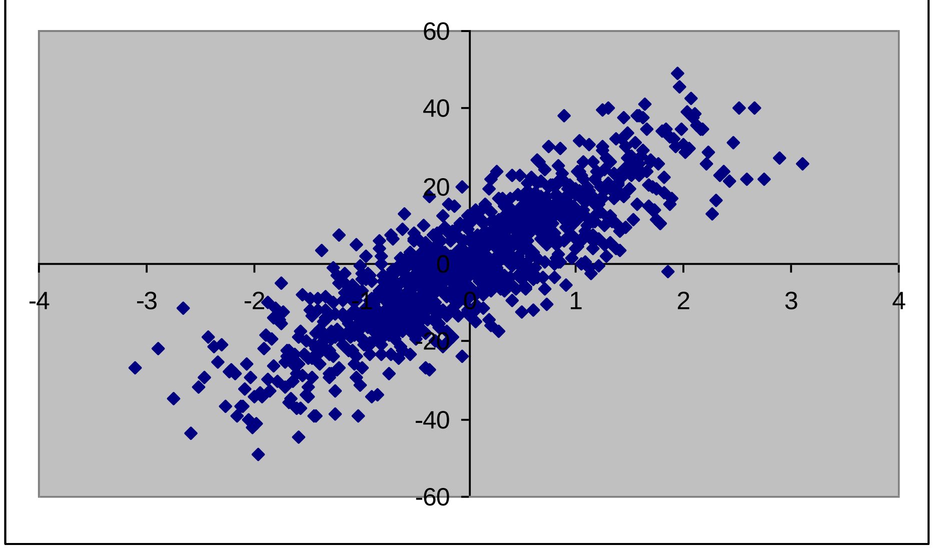
x_{N1} x_{N1} x_{N1}

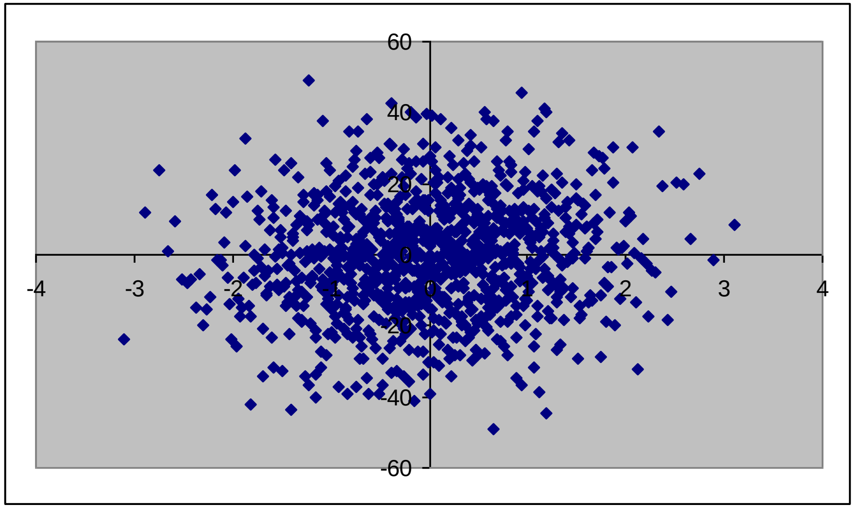
y_1

y_2

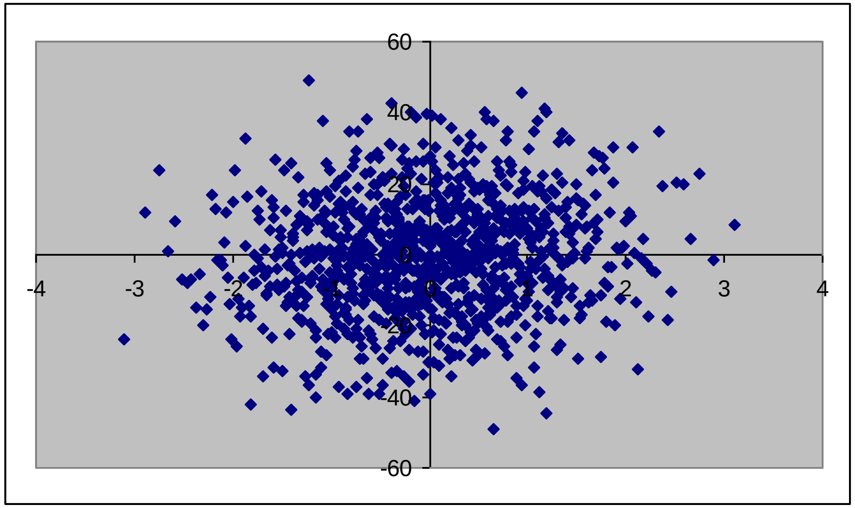
...

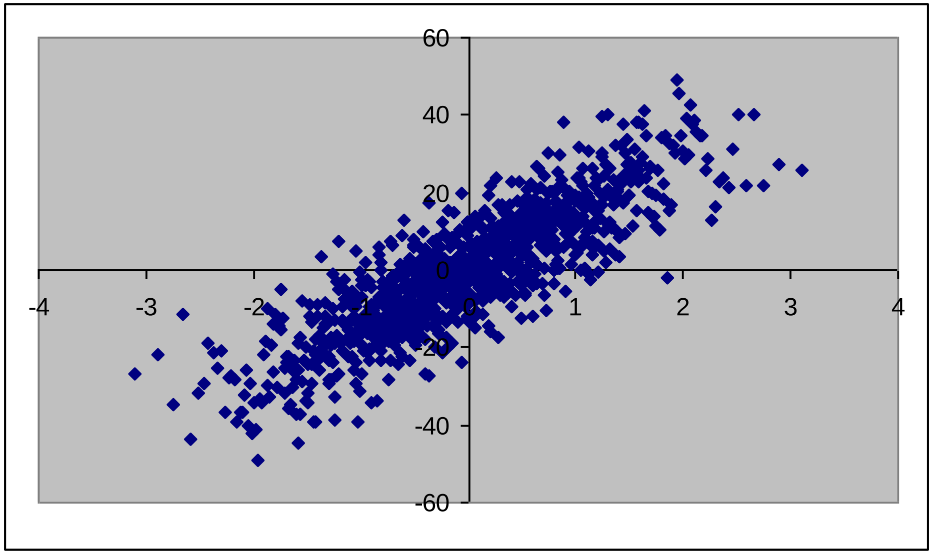
y_N





Scatterplots of y versus sorted factors

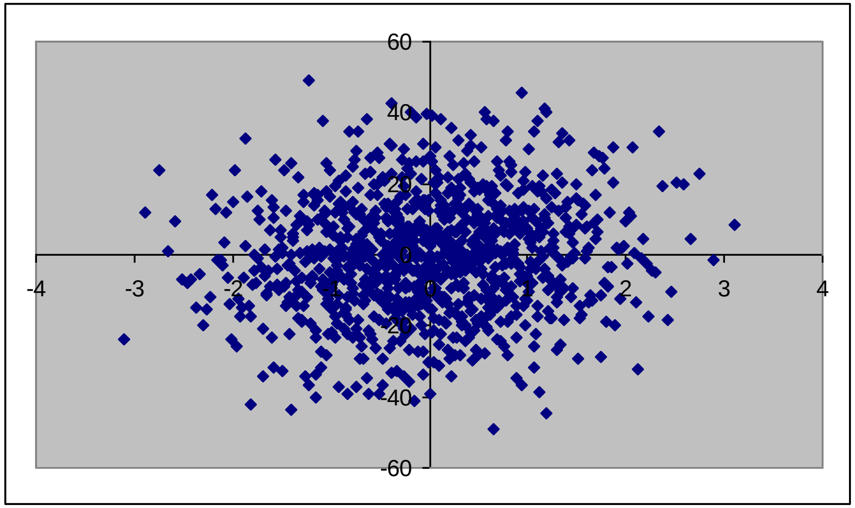


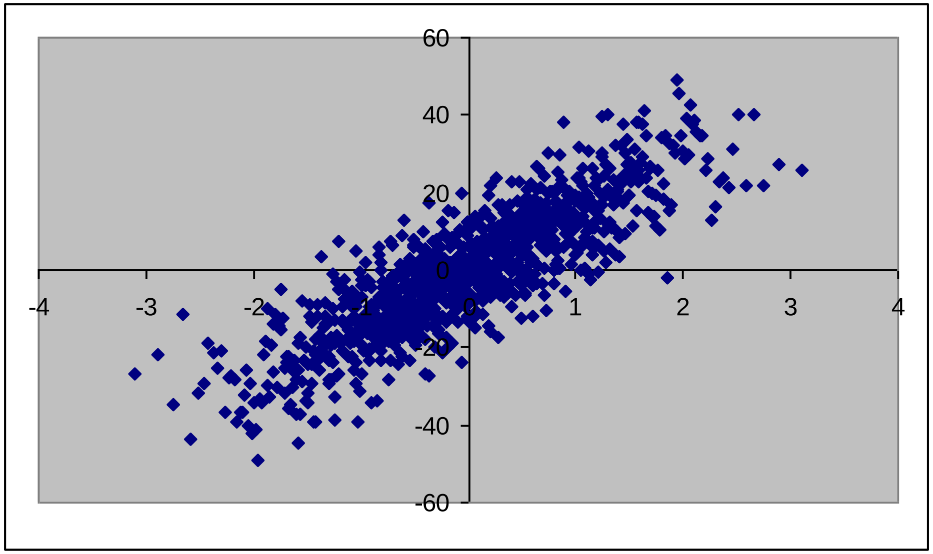


The ordinate axis is always Y

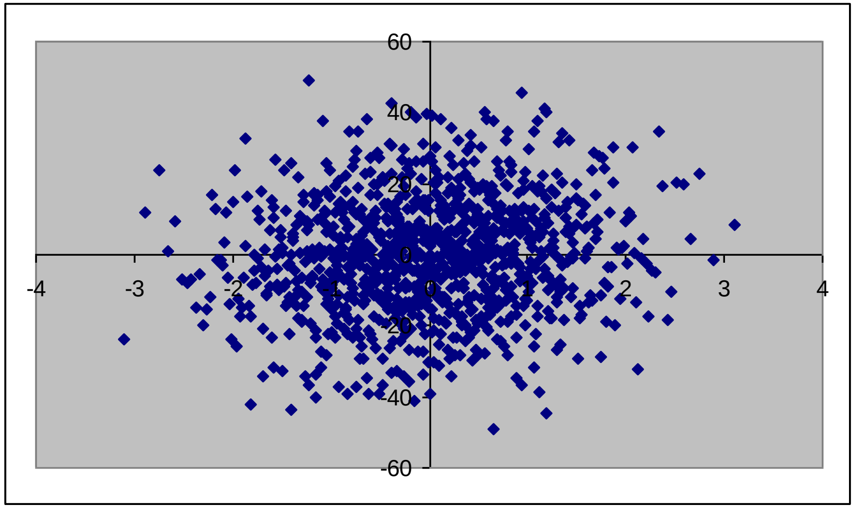
The abscissa are the various factors X_i in turn.

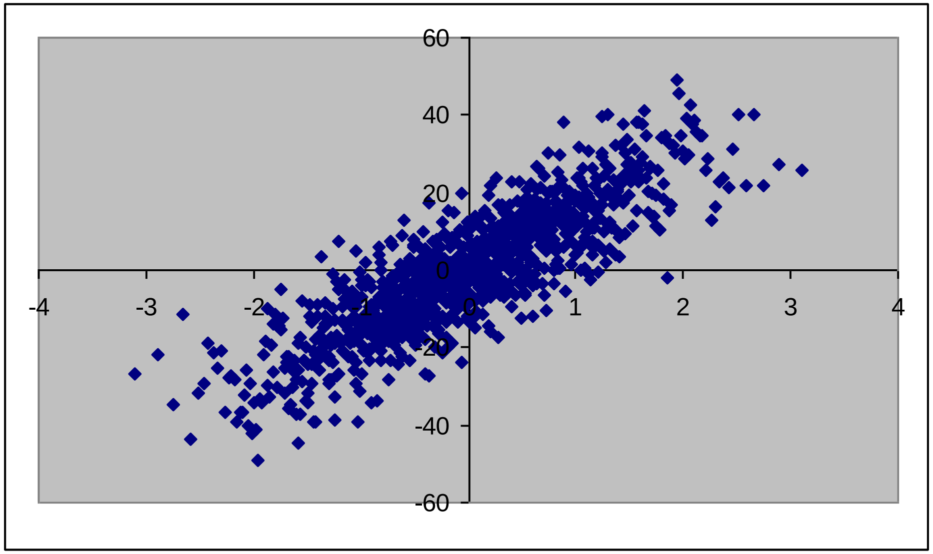
The points are always the same





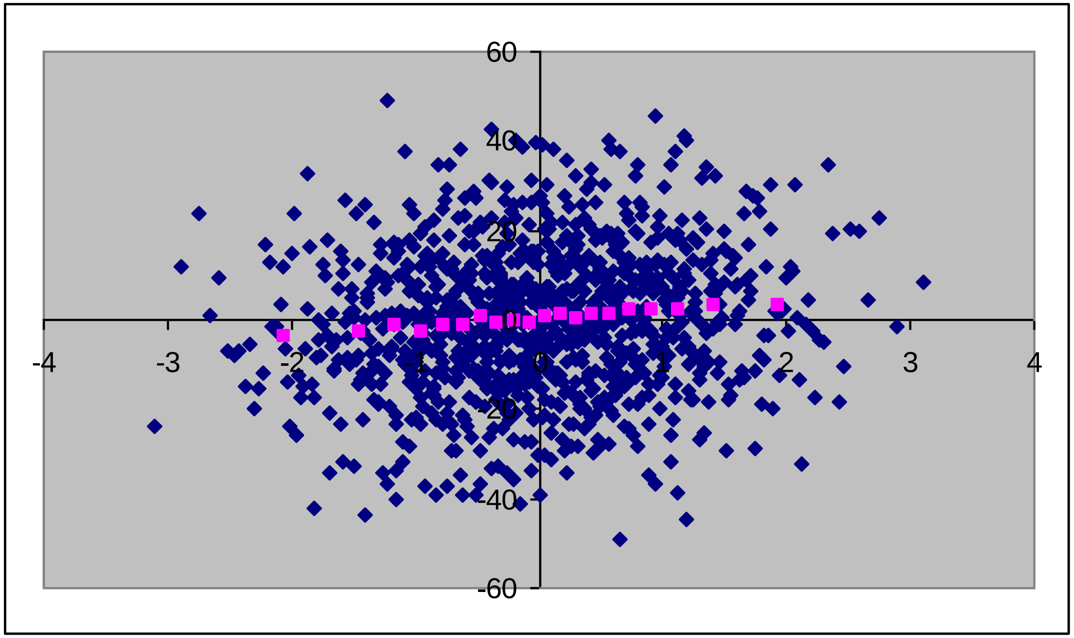
Which factor is more important?

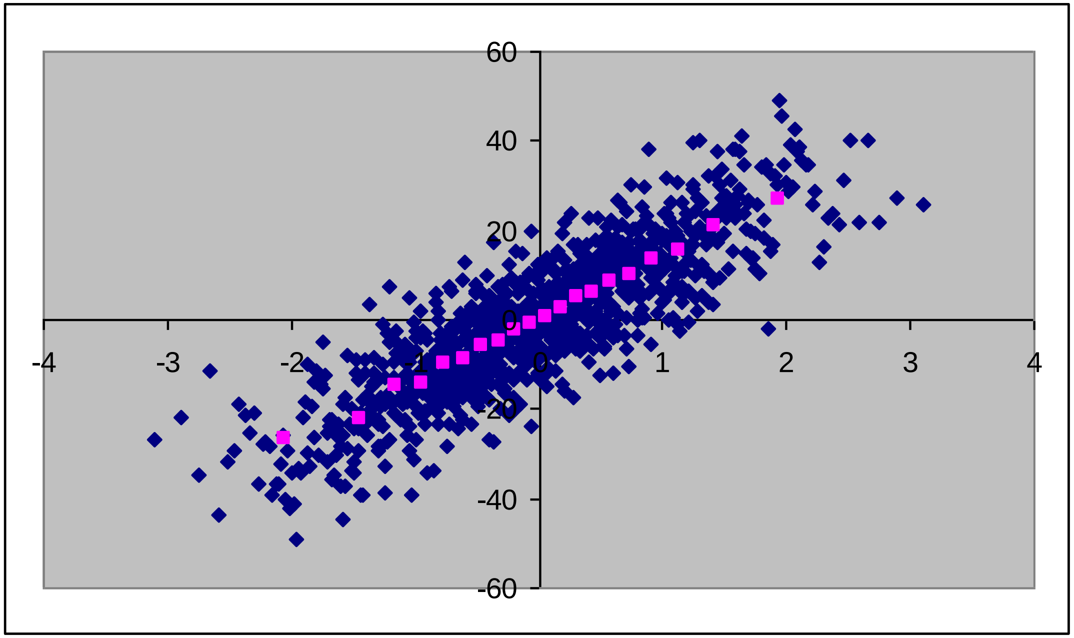




These are ~1,000 points

Divide them in 20 bins of ~ 50 points

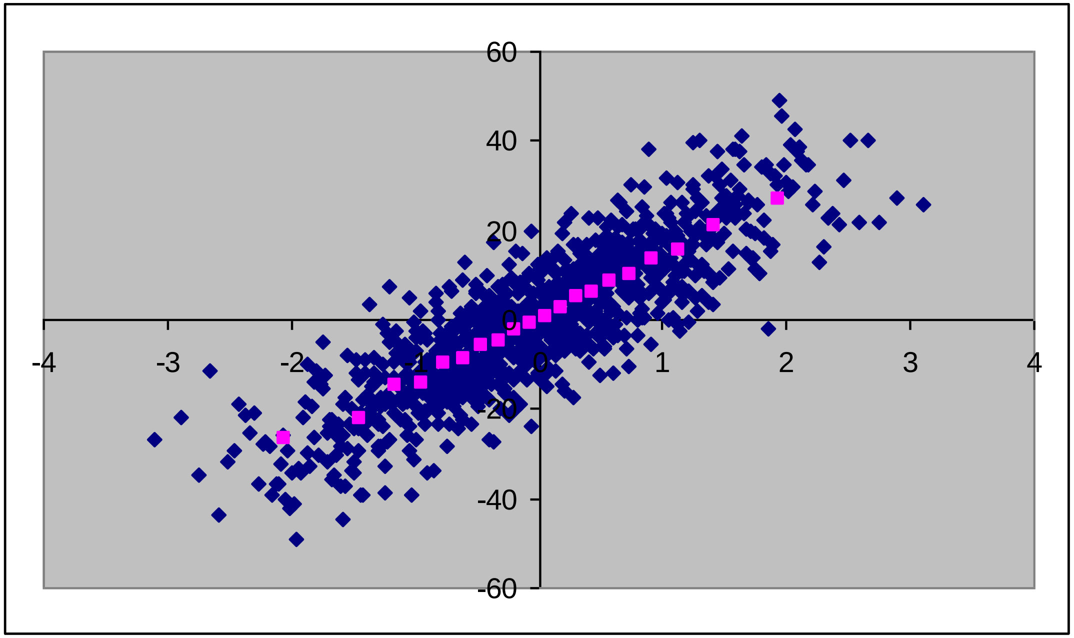




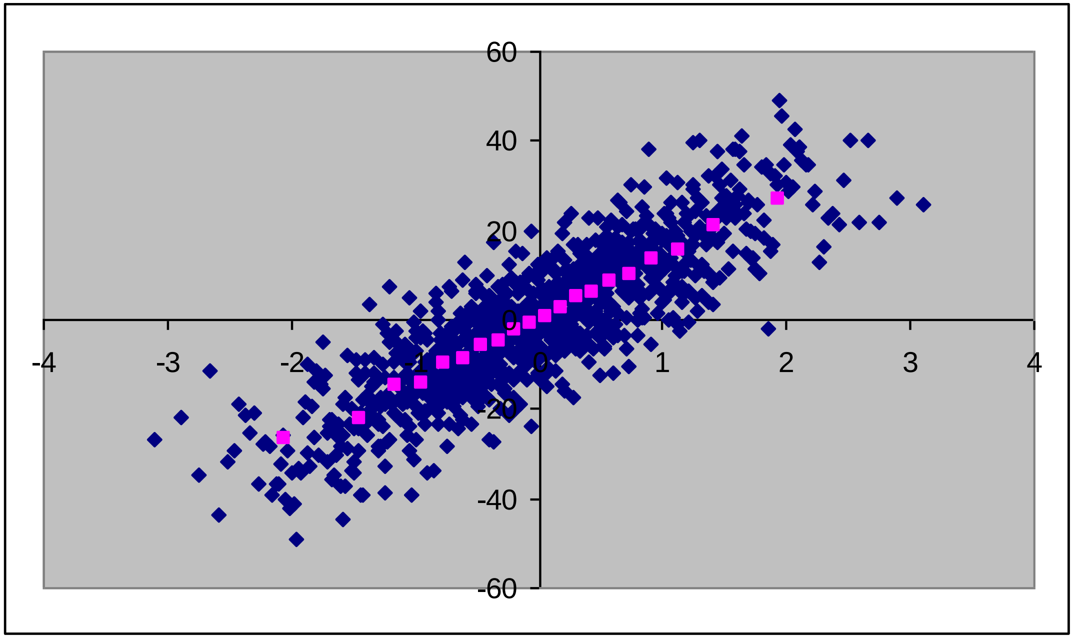
~1,000 blue points

Divide them in 20 bins of ~ 50 points

Compute the bin's average (pink dots)



Each pink point is ~ $$E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)$$



Take the variance of the pinkies

$$V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right)$$

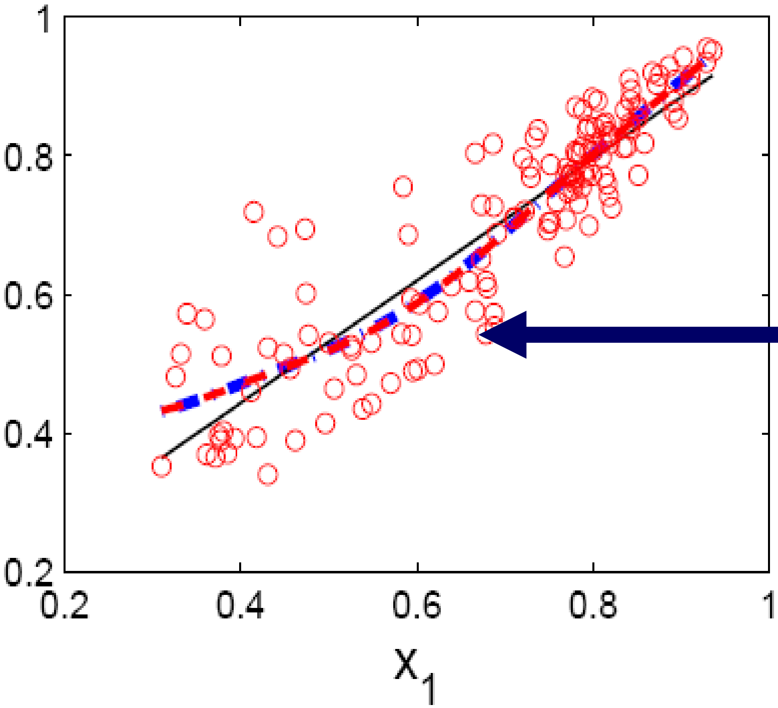
Pearson's correlation
ratio

Smoothed curve

$$S_i \equiv \eta_i^2 := \frac{V_{x_i}\left(E_{\mathbf{x}_{\sim i}}\left(y \mid x_i\right)\right)}{V\left(y\right)}$$

First order sensitivity index

Unconditional
variance



Smoothed curve

$$\mathrm{E}_{\mathbf{x}_{\sim i}}\left(y \mid x_i\right)$$

First order sensitivity index:

$$\frac{\mathrm{V}_{x_i}\left(\mathrm{E}_{\mathbf{x}_{\sim i}}\left(y \mid x_i\right)\right)}{\mathrm{V}\left(y\right)}$$

$$V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right)$$

First order effect, or top marginal variance=

= the expected reduction in variance than would be achieved if factor Xi could be fixed.

Why?

Because:

$$V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right)+$$

$$+E_{X_i}\left(V_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right)=V(Y)$$

Easy to prove using $V(\bullet)=E(\bullet)2-E2(\bullet)$

Because:

$$V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right) + \boxed{E_{X_i}\left(V_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right)} = V(Y)$$

This is what variance would be left (on average) if Xi could be fixed…

… then this …

$$\boxed{V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \mid X_i\right)\right)} +$$

$$+ E_{X_i}\left(V_{\mathbf{X}_{\sim i}}\left(Y \mid X_i\right)\right) = V(Y)$$

… must be the expected reduction in variance than would be achieved if factor Xi could be fixed

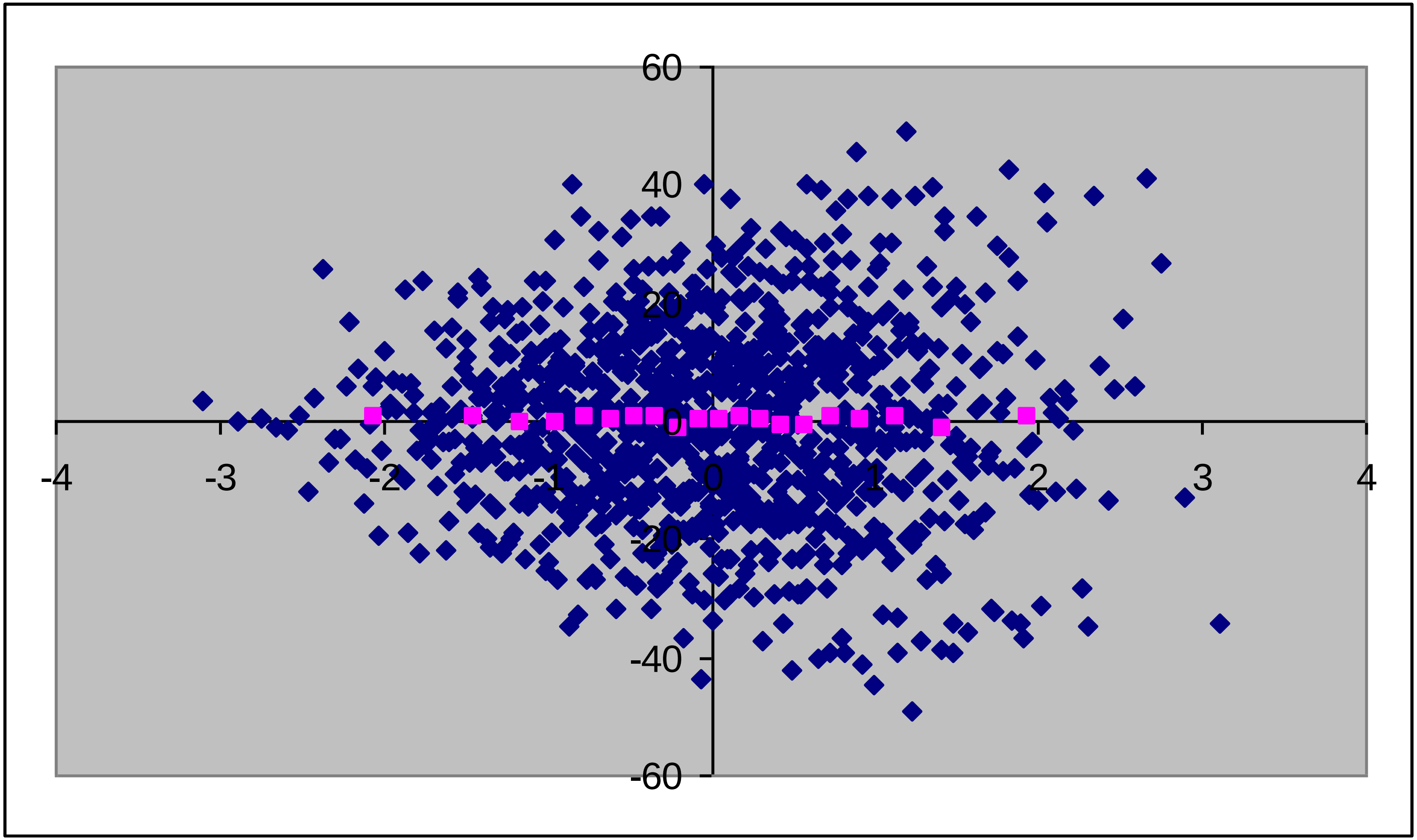
For <u>additive</u> models one can decompose the total variance as a sum of first order effects

$$\sum_i V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right) \approx V(Y)$$

… which is also how additive models are defined

How about non additive models?

- Is $S_i = 0$?
- Is this factor non-important?



There are ‘importance’ terms which capture two-way, three way, … interactions among variables … as in experimental design…

… and all these terms are linked by a formula

Variance decomposition (ANOVA)

$$V(Y) =$$

$$\sum_i V_i + \sum_{i,j>i} V_{ij} + \ldots + V_{123\ldots k}$$

Variance decomposition (ANOVA)

$$V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right) = V_i$$

$$V_{X_i X_j}\left(E_{\mathbf{X}_{\sim ij}}\left(Y \middle| X_i X_j\right)\right) =$$

$$= V_i + V_j + V_{ij}$$

…

Variance decomposition (ANOVA)

When the factors are independent the total variance can be decomposed into main effects and interaction effects up to the order k, the dimensionality of the problem.

Variance decomposition (ANOVA)

When the factors are <u>not</u> independent the decomposition loses its unicity (and hence its appeal!)

If fact interactions terms are awkward to handle: second order terms are as many as $k(k-1)/2 \cdots$

Wouldn’t it be handy to have just a single ‘importance’ terms for all effects, inclusive of first order and interactions?

In fact such terms exist and can be computed easily, without knowledge of the individual interaction terms

Thus given a model $Y=f(X_1,X_2,X_3)$

Instead of

$$V=V_1+V_2+V_3+$$
$$+V_{12}+V_{13}+V_{23}+$$
$$+V_{123}$$

and

$$1=S_1+S_2+S_3+$$
$$+S_{12}+S_{13}+S_{23}+$$
$$+S_{123}$$

We have:

$$S_{T1}=S_1+S_{12}+S_{13}+S_{123}$$

(and analogue formulae for S_{T2}, S_{T3}) which can be computed without knowing S_1, S_{12}, S_{13}, S_{123}

S_{T1} is called a total effect sensitivity index

How to get from first order to total order

From

$$V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right)$$

Main effect of factor X_i

replacing X_i with $\mathbf{X}_{\sim i}$

To main effect of non-X_i

$$V_{\mathbf{X}_{\sim i}}\left(E_{X_i}\left(Y \middle| \mathbf{X}_{\sim i}\right)\right)$$

BUT:

$$V_{\mathbf{X}_{\sim i}}\left(E_{X_i}\left(Y\middle|\mathbf{X}_{\sim i}\right)\right)+$$

$$E_{\mathbf{X}_{\sim i}}\left(V_{X_i}\left(Y\middle|\mathbf{X}_{\sim i}\right)\right)=V(Y)$$

If this is the main effect on non-X_i…

$$E_{\mathbf{X}_{\sim i}}\left(V_{X_i}\left(Y \middle| \mathbf{X}_{\sim i}\right)\right)$$

$$V_{\mathbf{X}_{\sim i}}\left(E_{X_i}\left(Y \middle| \mathbf{X}_{\sim i}\right)\right)$$

… all remaining variance
must be due to X_i and its
interactions

Main effects

Residuals

$$V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right) \qquad E_{X_i}\left(V_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right)$$

$$V_{\mathbf{X}_{\sim i}}\left(E_{X_i}\left(Y \middle| \mathbf{X}_{\sim i}\right)\right) \qquad E_{\mathbf{X}_{\sim i}}\left(V_{X_i}\left(Y \middle| \mathbf{X}_{\sim i}\right)\right)$$

Main (or first order) effect of X_i

Main effects

Residuals

$$\boxed{V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \mid X_i\right)\right)} + E_{X_i}\left(V_{\mathbf{X}_{\sim i}}\left(Y \mid X_i\right)\right) = \mathbf{V(Y)}$$

$$V_{\mathbf{X}_{\sim i}}\left(E_{X_i}\left(Y \mid \mathbf{X}_{\sim i}\right)\right) + \boxed{E_{\mathbf{X}_{\sim i}}\left(V_{X_i}\left(Y \mid \mathbf{X}_{\sim i}\right)\right)} = \mathbf{V(Y)}$$

Total (or total order) effect of X_i

$$E_{\mathbf{X}_{\sim i}}\left(V_{X_i}\left(Y|\mathbf{X}_{\sim i}\right)\right)$$

Total effect, or bottom marginal variance=

= the expected variance than would be left if all factors but Xi could be fixed.

$$\frac{V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right)}{V(Y)} = S_i$$

$$\frac{E_{\mathbf{X}_{\sim i}}\left(V_{X_i}\left(Y \middle| \mathbf{X}_{\sim i}\right)\right)}{V(Y)} = S_{Ti}$$

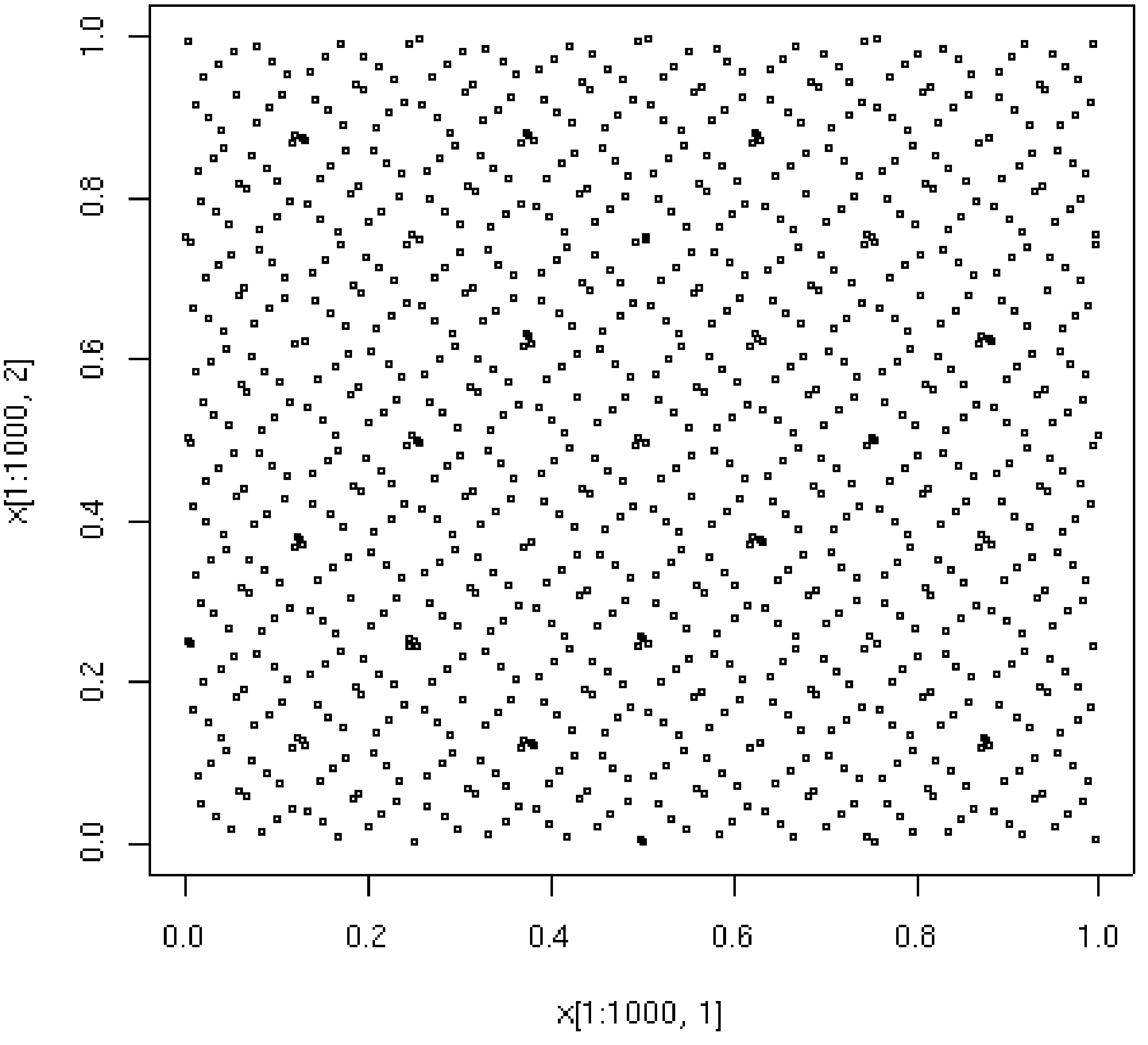
Rescaled to [0,1], under the name of first order and total order sensitivity coefficient

Variance based measures are:
-well scaled,
-concise,
-easy to communicate.

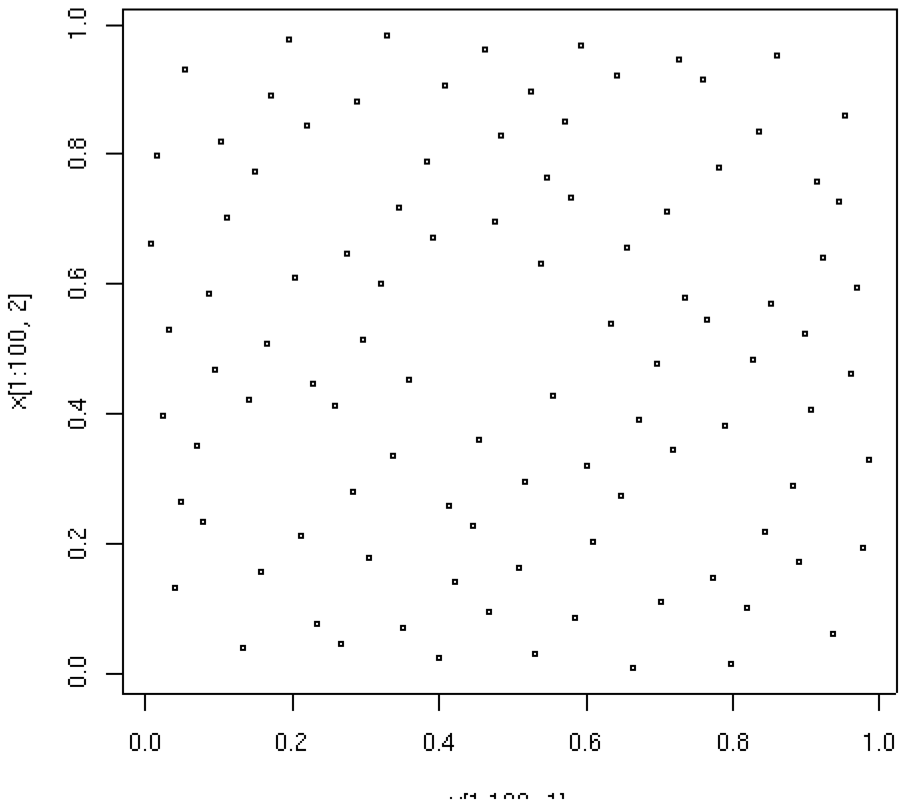
Further
- S_i reduces to squared standard regression coefficients for linear model.
- S_{Ti} detect and describe interactions and
- Becomes a screening test at low sample size.

Both indices can be computed via Monte Carlo

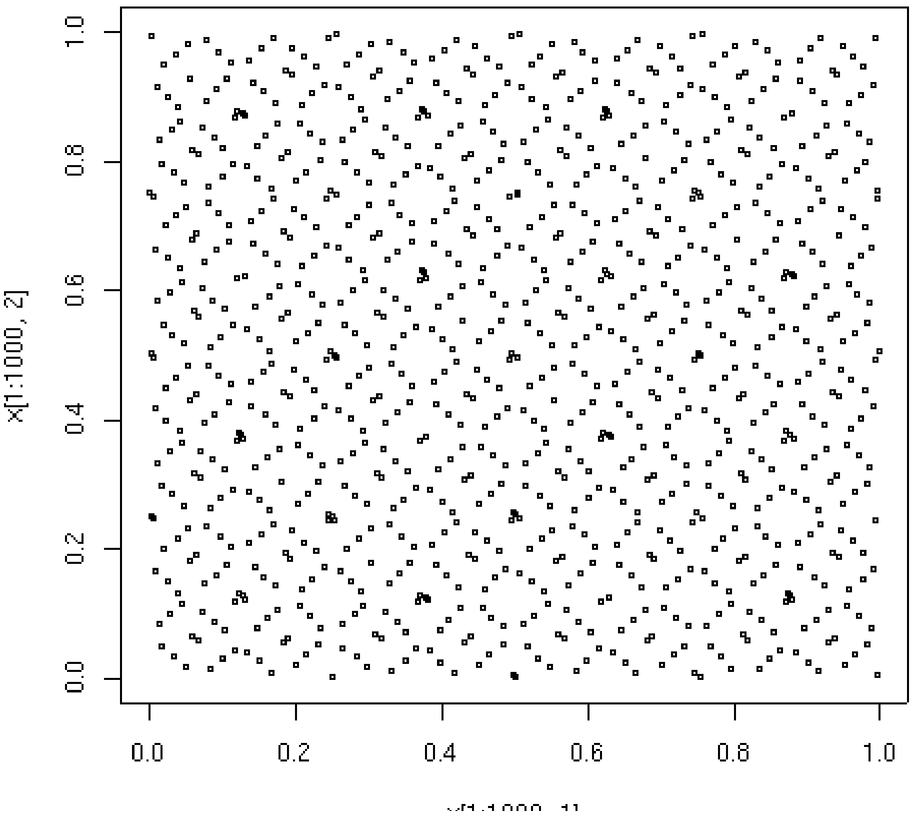
We use quasi random sequences developed by I.M. Sobol’



An LP_τ sequence

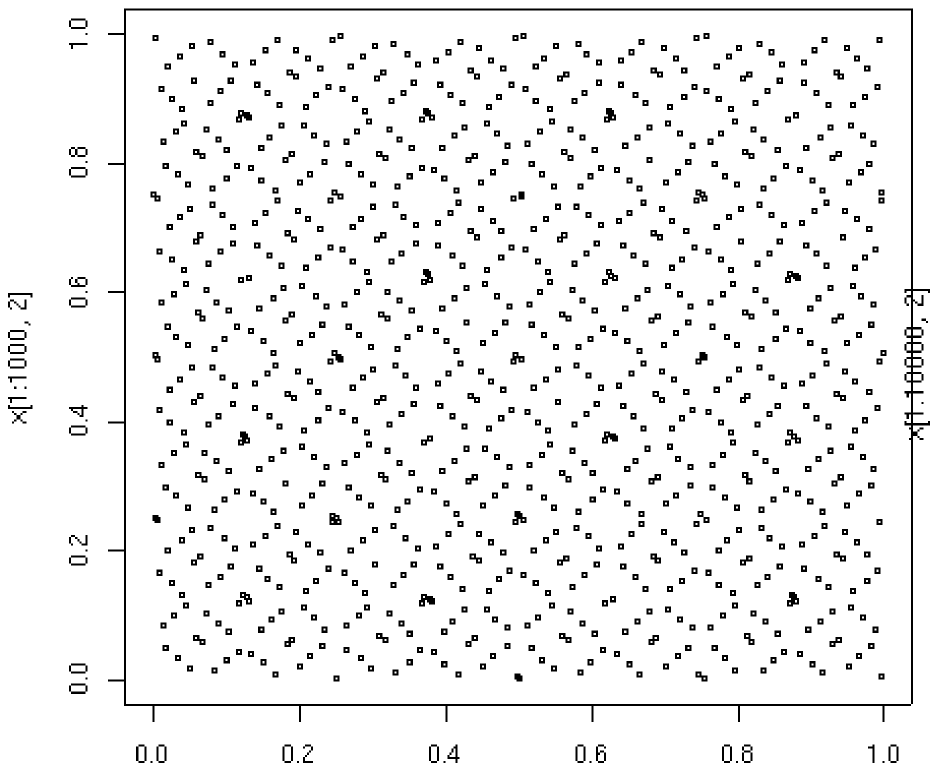


X1,X2 plane, 100 Sobol' points

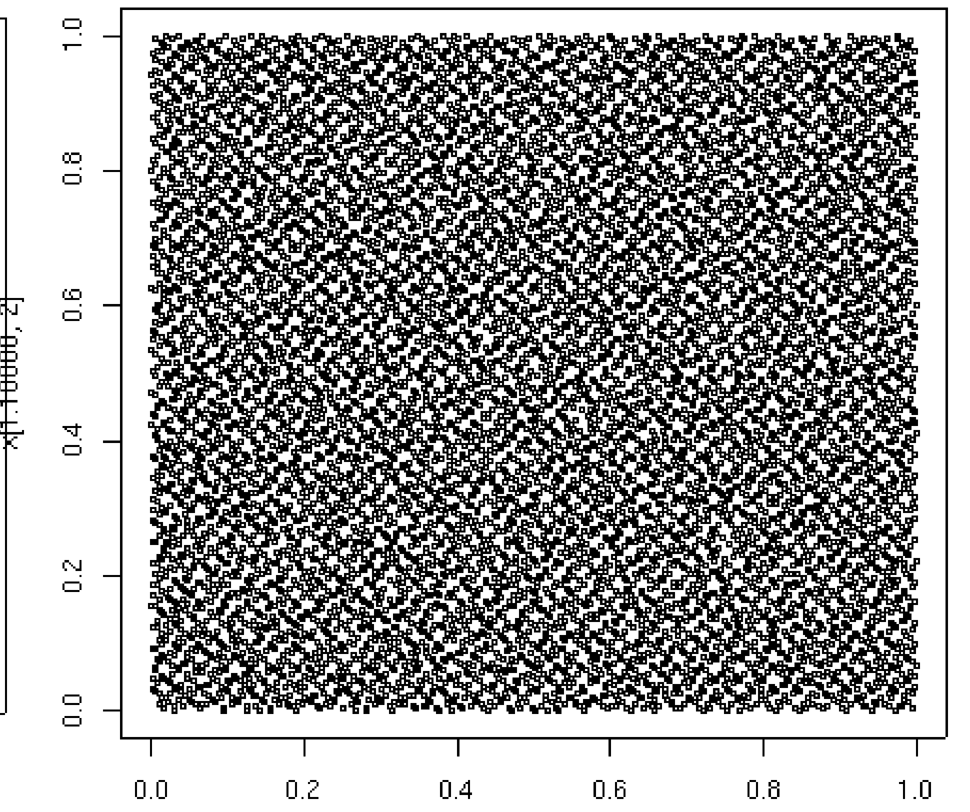


X1,X2 plane, 1000 Sobol' points

Sobol' sequences of quasi-random points

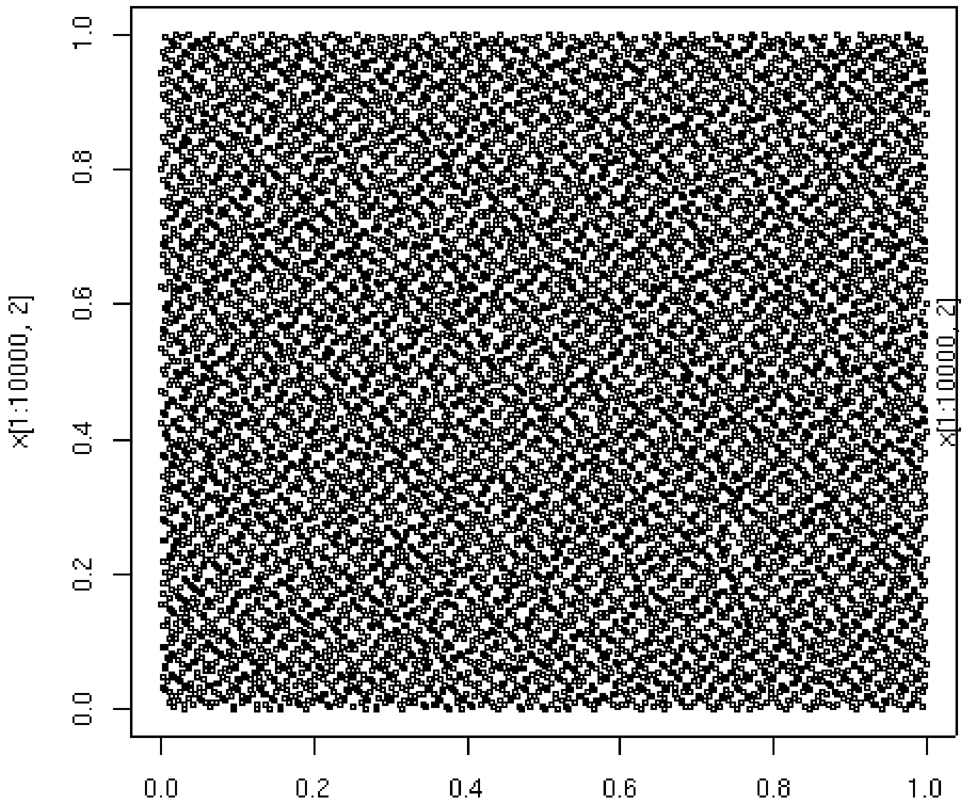


X1,X2 plane, 1000 Sobol' points

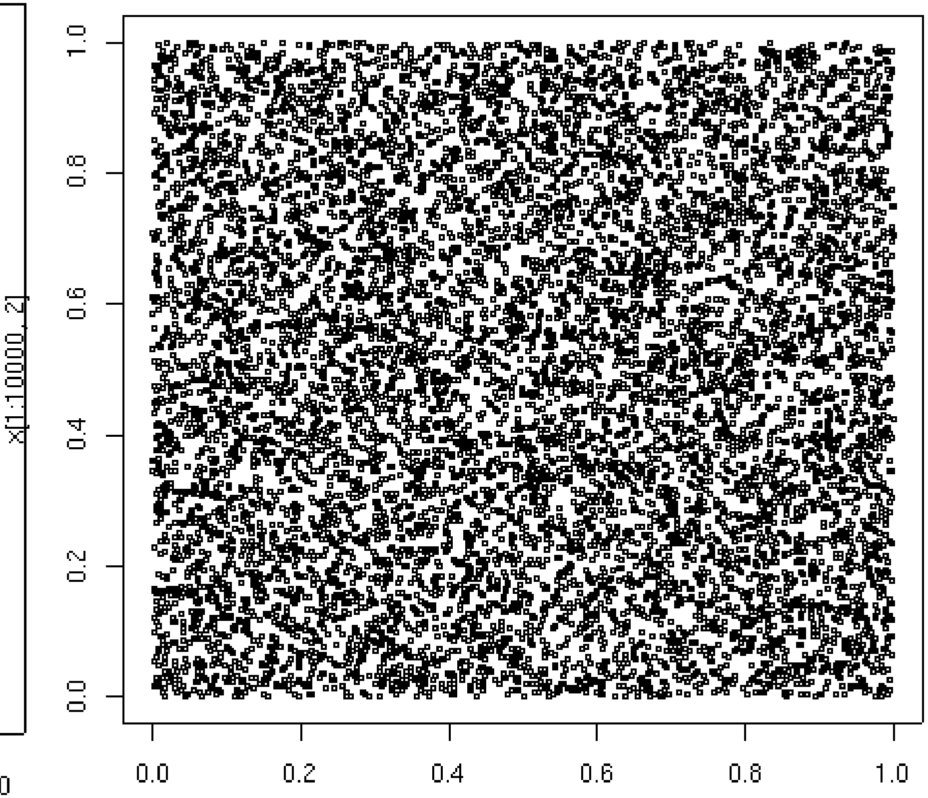


X1,X2 plane, 10000 Sobol' points

Sobol' sequences of quasi-random points



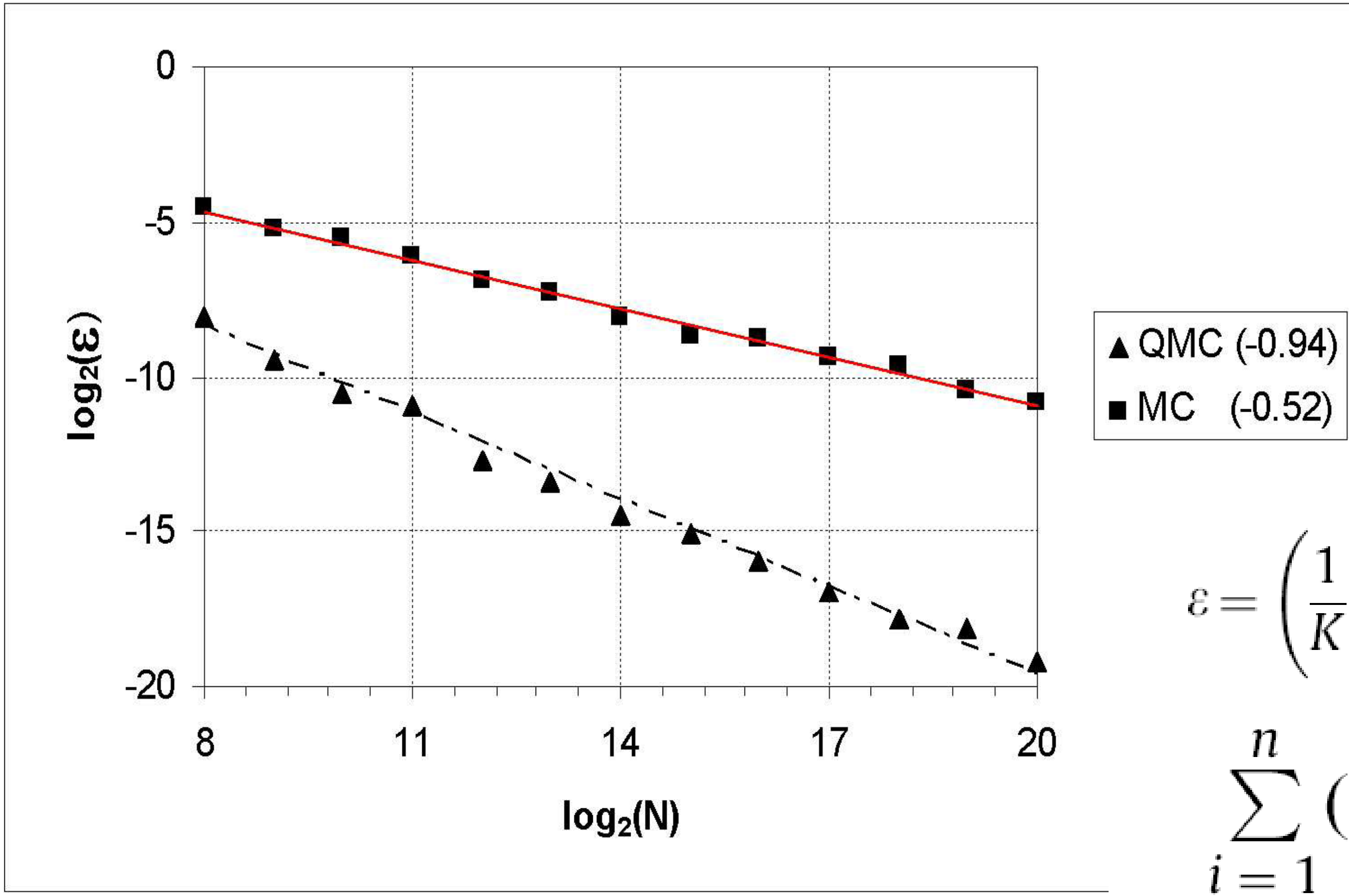
X1,X2 plane, 10000 Sobol' points



X1,X2 plane, 10000 random points

Sobol' sequences of quasi-random points against random points

Why quasi-random



Sergei Kucherenko, Imperial College London

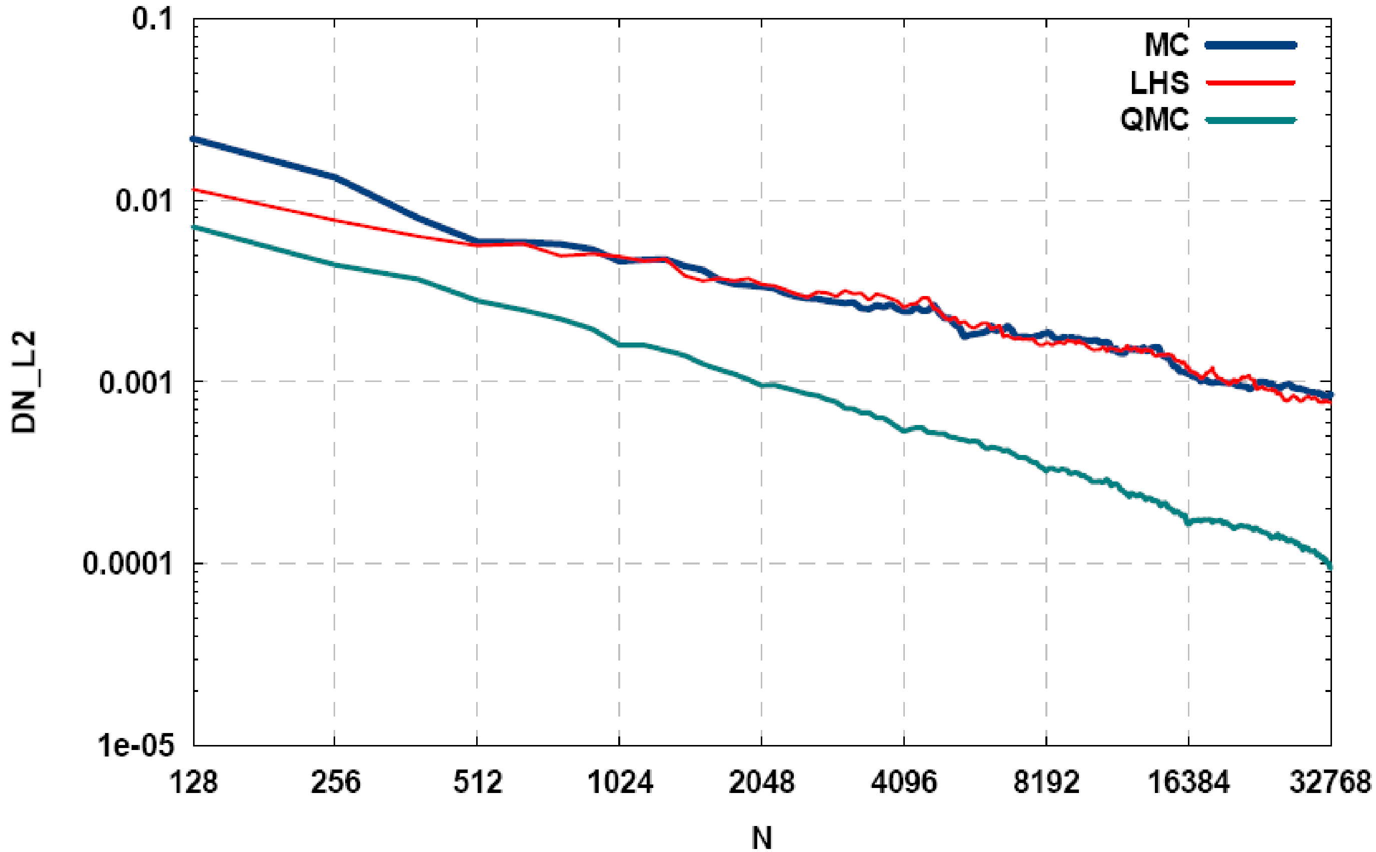
$$\varepsilon = \left(\frac{1}{K} \sum_{k=1}^{K} \left(I[f] - I_k[f] \right)^2 \right)^{1/2}$$

$$\sum_{i=1}^{n} (-1)^i \prod_{j=1}^{i} x_j$$

Root mean square error over 50 different trials. The error refers to the numeric-versus-analytic value the integral of the function over its dominion.

Source: Kucherenko S., Feil B., Shah N., Mauntz W. The identification of model effective dimensions using global sensitivity analysis Reliability Engineering and System Safety 96 (2011) 440-449.

Quasi-random versus Latin Hypercube



(a)

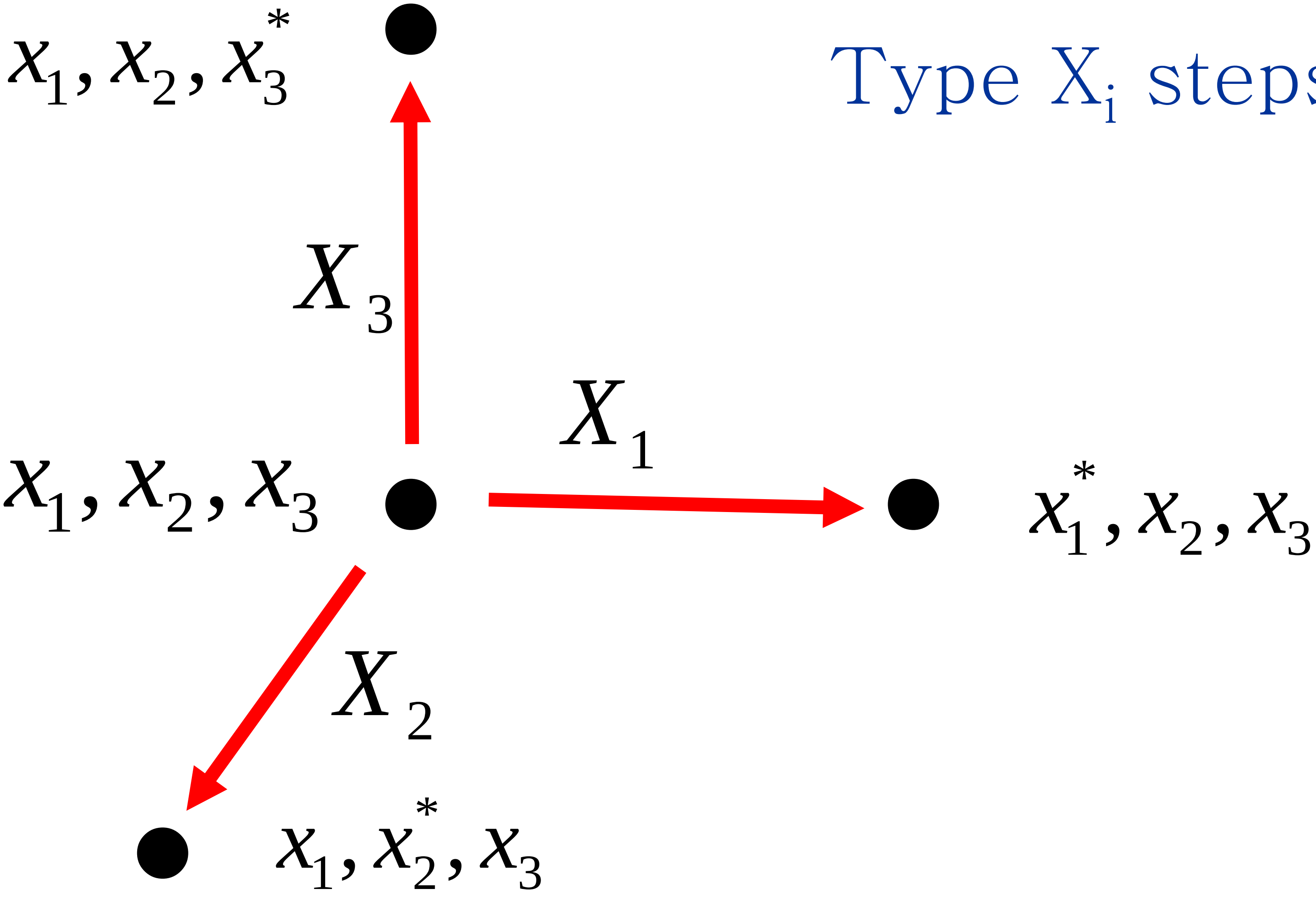
Discrepancy at different sample sizes. Monte Carlo versus Quasi Monte Carlo and Latin Hypercube Sampling.

Source: Kucherenko, S., Albrecht, D., Saltelli, A., 2015, Exploring multi-dimensional spaces: a Comparison of Latin Hypercube and Quasi Monte Carlo Sampling Techniques, Submitted to SIAM/ASA Journal on Uncertainty Quantification.

How to use the MC points?

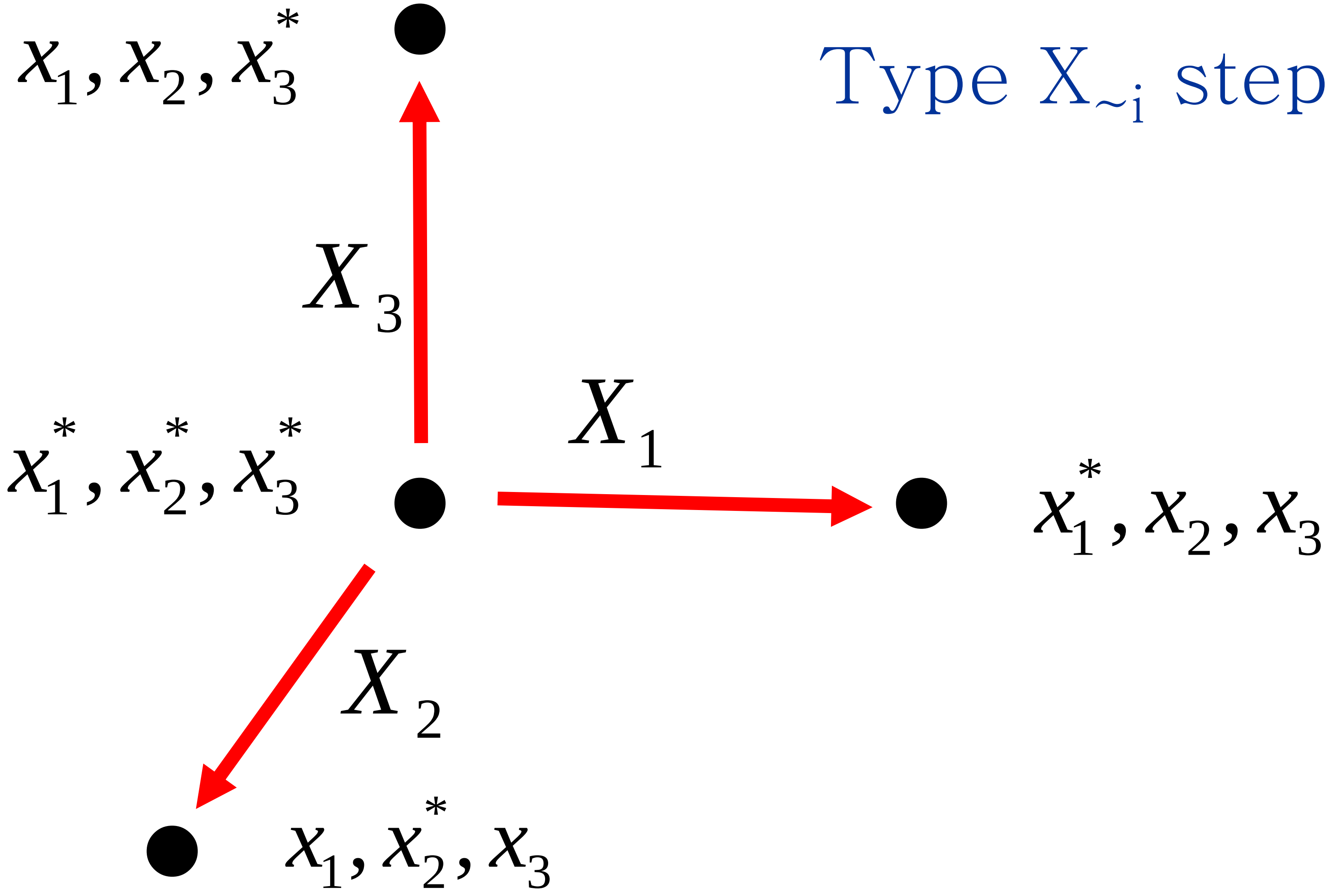
Computing S_{Ti}

Type X_i steps



Computing S_i

Type $X_{\sim i}$ steps



Estimation procedures!

- No brute force. It is not needed to use a double loop, though the measures are expresses as V(E(•)) and E(V(•)).
- For S_i quick estimation procedures are available which are k-independent.
- For S_{Ti} estimation procedures are mostly k-dependent (unless … active area of research…).

Why these measures?

$$V_{X_i}\left(E_{\mathbf{X}_{\sim i}}\left(Y \middle| X_i\right)\right)$$

Factors prioritization

$$E_{\mathbf{X}_{\sim i}}\left(V_{X_i}\left(Y \middle| \mathbf{X}_{\sim i}\right)\right)$$

Fixing (dropping) non important factors

Saltelli A. Tarantola S., 2002, On the relative importance of input factors in mathematical models: safety assessment for nuclear waste disposal, *Journal of American Statistical Association*, **97** (459), 02-709.

More about the settings:

• Factor prioritisation → $S_i \equiv \dfrac{V\left(E\left(Y \middle| X_i\right)\right)}{V_Y}$

If the cost of 'discovering' factors were the same for all factors which factor should I try to discover first?

•Factor fixing: Can I fix a factor [or a subset of input factors] at any given value over their range of uncertainty without reducing significantly the output?

$$S_{Ti} \equiv \frac{E\left(V\left(Y \middle| \mathbf{X}_{\sim i}\right)\right)}{V_Y}$$

Factor fixing is useful to achieve model simplification and 'relevance'.

We cannot use S_i to fix a factor; $S_i = 0$ is a necessary condition for X_i to be non-influential but not a sufficient one

X_i could be influent at the second order.

Factor fixing

Imagine that X_i is non-influential and we compute:

$$V_{X_i}\left(Y \mid \mathbf{X}_{\sim i} = \mathbf{x}^*_{\sim i}\right)$$

Non-X_i fixed to a point in the non-X_i space

But this must be zero because if X_i is non-influential than all depends from $X_{\sim i}$ and fixing it freezes the variance. Then:

$$E\left(V\left(Y \mid \mathbf{X}_{\sim i}\right)\right) = 0$$

and S_{Ti} is zero

Factor fixing

We have just proven that if X_i is non-influential then S_{Ti} is zero (necessary condition). Conversely if S_{Ti} is zero then

$$E\left(V\left(Y \mid \mathbf{X}_{\sim i}\right)\right) = 0$$

By definition. But a variance can only be a positive number and if an average of variances is zero than all variances must be zero, which proves that:

$$V\left(Y \mid \mathbf{X}_{\sim i} = \mathbf{x}^{*}_{\sim i}\right)$$

is also zero for any value of the fixed point $\mathbf{x}^{*}_{\sim i}$. This proves that nowhere in the space of $X_{\sim i}$ the factor Xi has any influence (sufficient condition).

Remarks on factor fixing: 1

Model simplification supported by factor fixing is useful.

"As the complexity of a system increases … precision and significance (or relevance) become almost mutually exclusive characteristics"

Zadeh's incompatibility principle (1965).

Lofti Zadeh

Remarks on factor fixing: 2

The model 'relevance' problem

$$R = \frac{\text{number of factors that truly induce variations in the output of interest}}{\text{total number of factors in the model}}$$

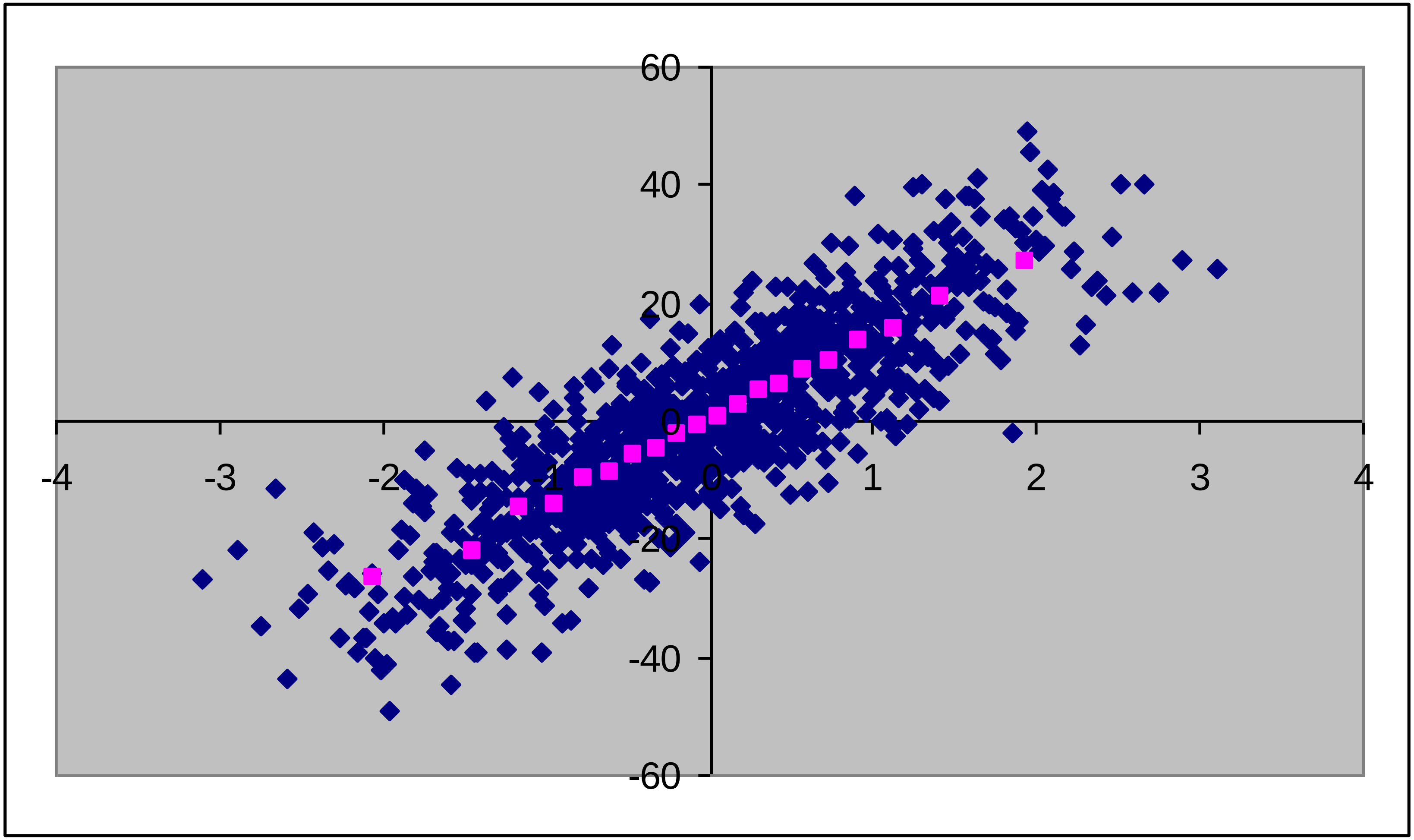
Low R could flag a model meant to intimidate.

Bruce Beck

Summary:

1. Easy-to-code, Monte Carlo – better on quasi-random points. Estimate of the error available.

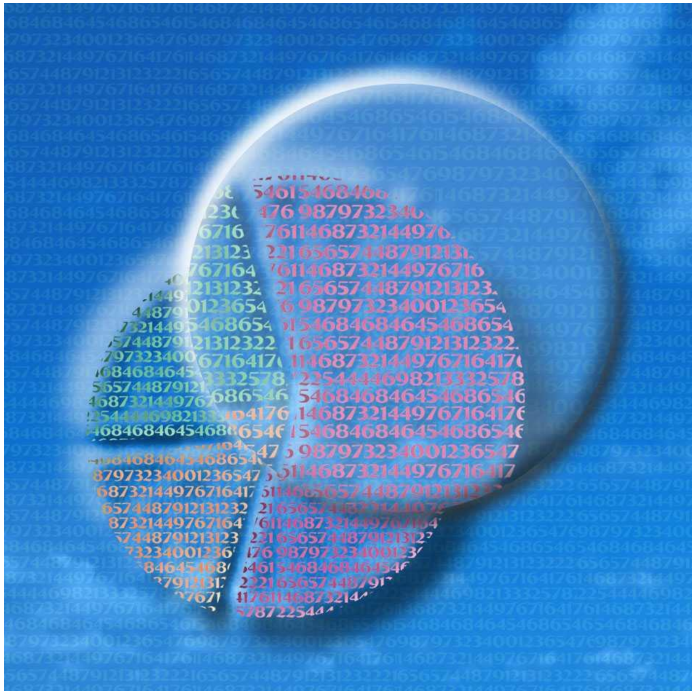
2. The main effect can be made cheap; its computational cost does not depend upon k.



Easy to smooth and interpolate!

Summary

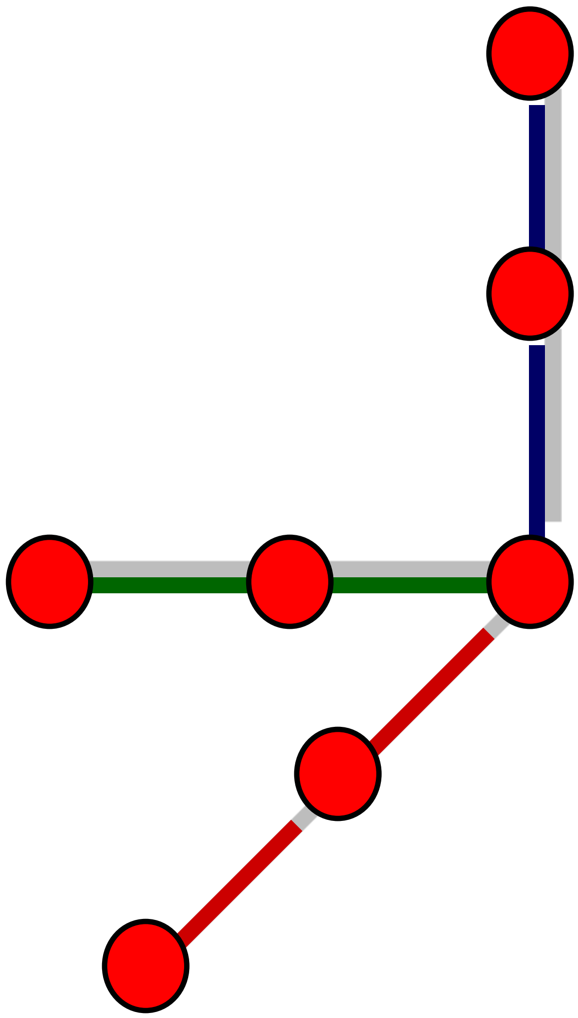
3. The total effect is more expensive; its computational cost is $(k+1)N$ where N is one of the order of one thousand.



Can something be done to ease adoption?

From:

Campolongo F, Saltelli A, Cariboni, J, 2011, From screening to quantitative sensitivity analysis. A unified approach, *Computer Physics Communication*, 182 (4), pp. 978–988.

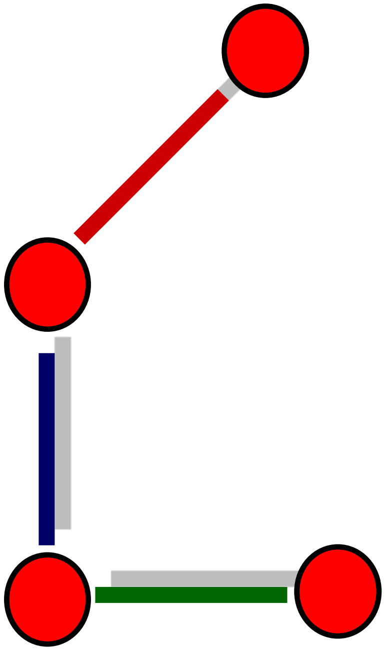


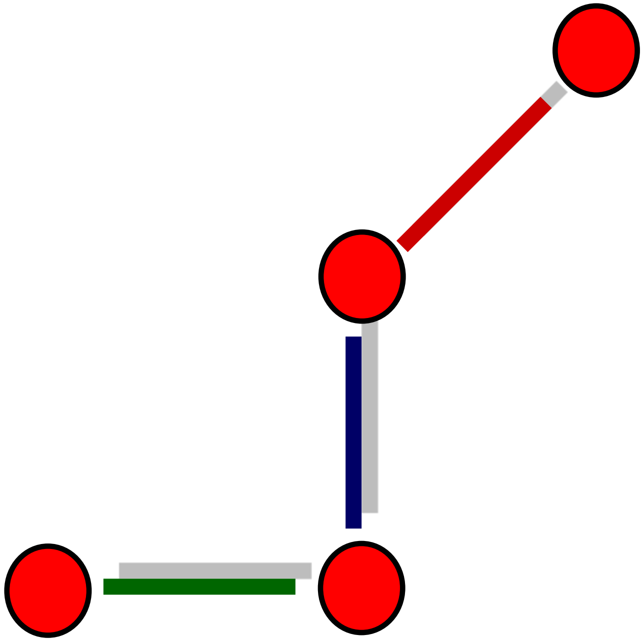
In 3 dimensions,
OAT, 7 points

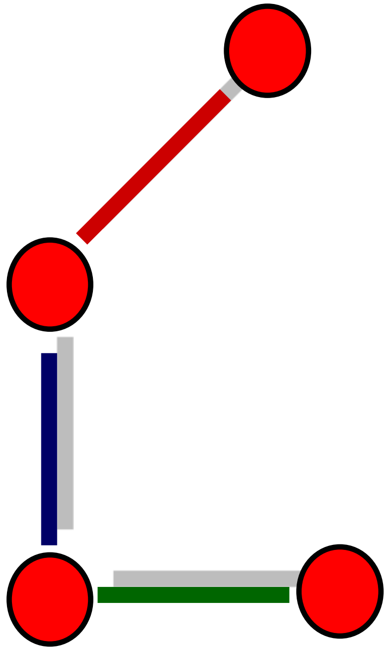
This is what is done

In 3 dimension, 8 screening points in a trajectory arrangement

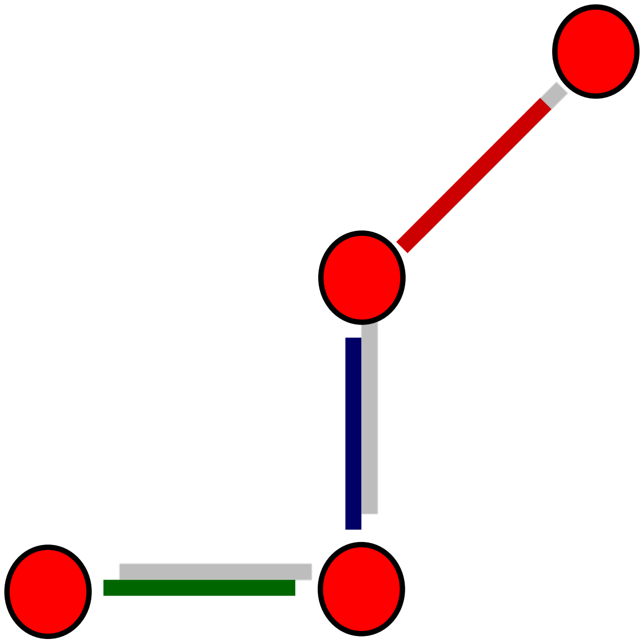
This is what could be done





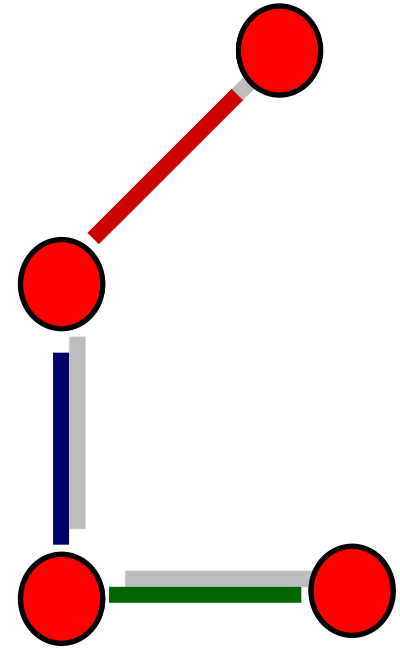


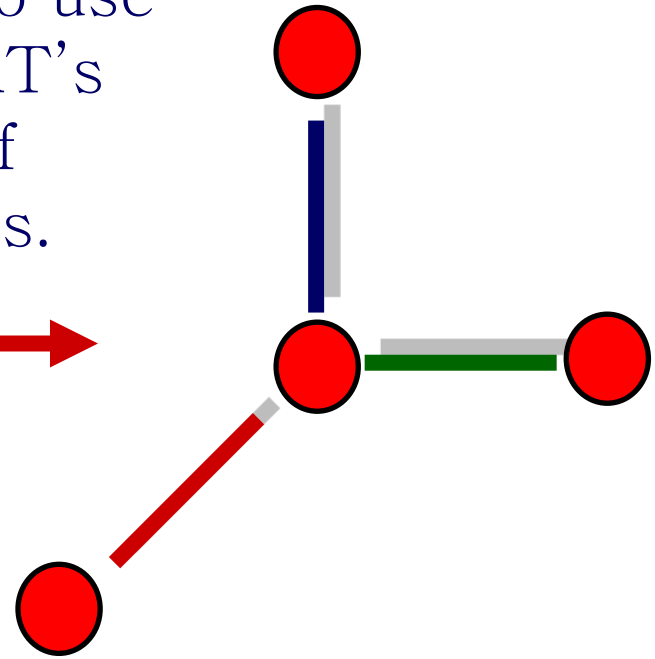
This is a screening good practice (Morris, or method of the elementary effects)

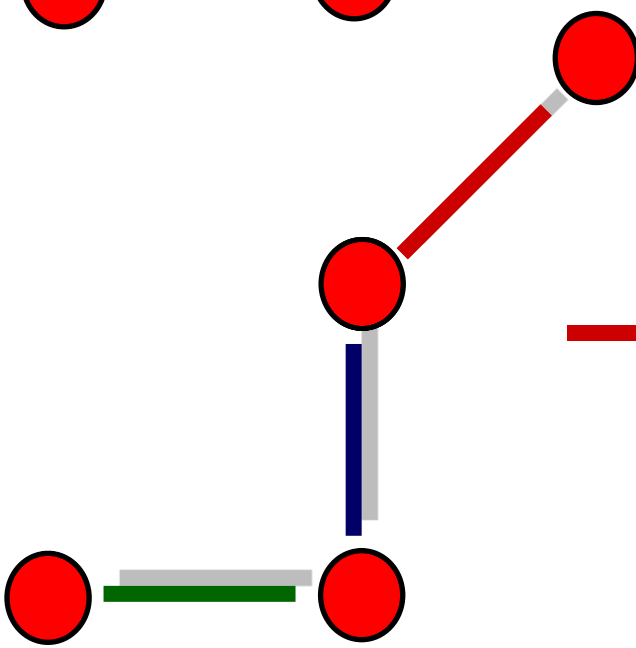


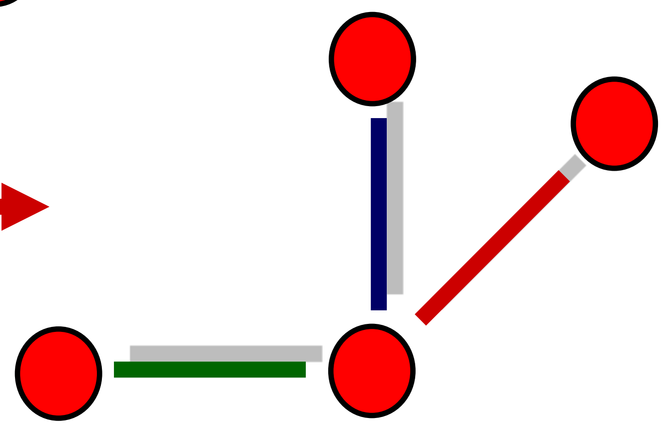
See: Campolongo, F., Cariboni, J., and Saltelli, A., 2007, An effective screening design for sensitivity analysis of large models, *Environmental Modelling and Software*, **22**,1509-1518.

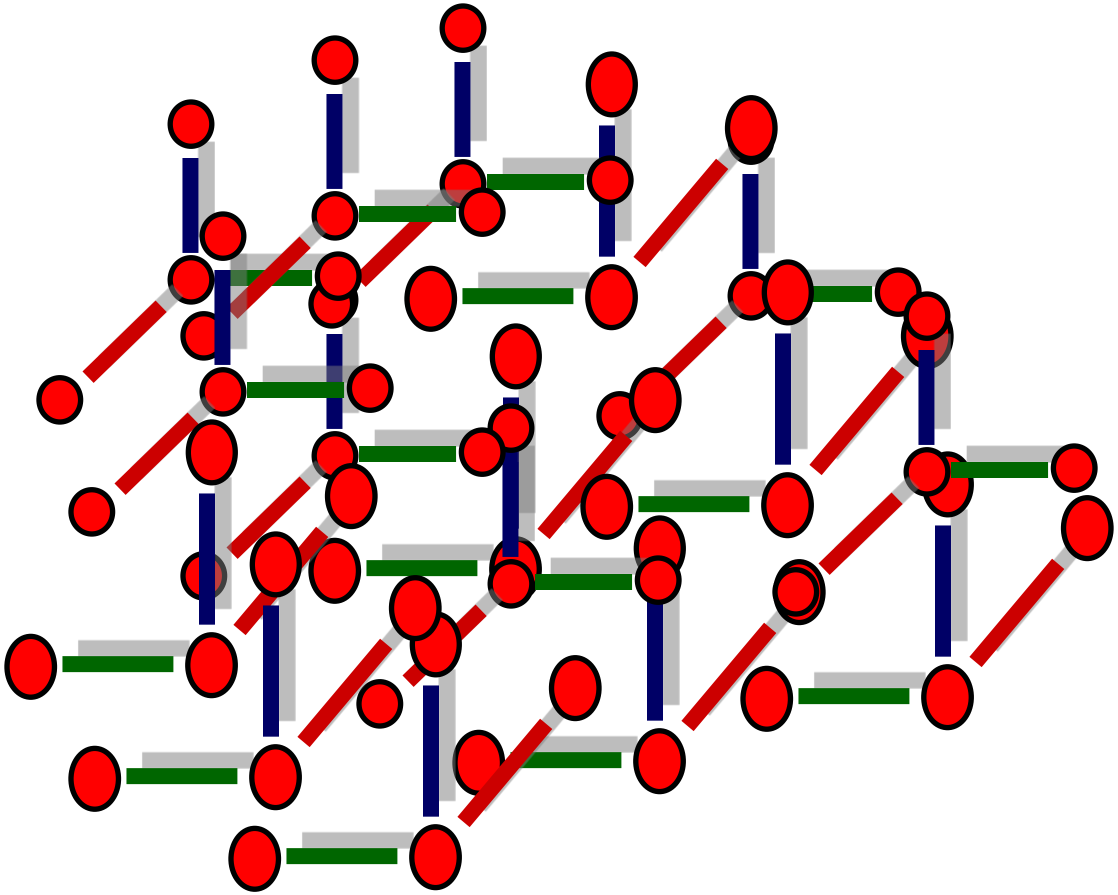
One could also use iterated OAT's instead of trajectories.





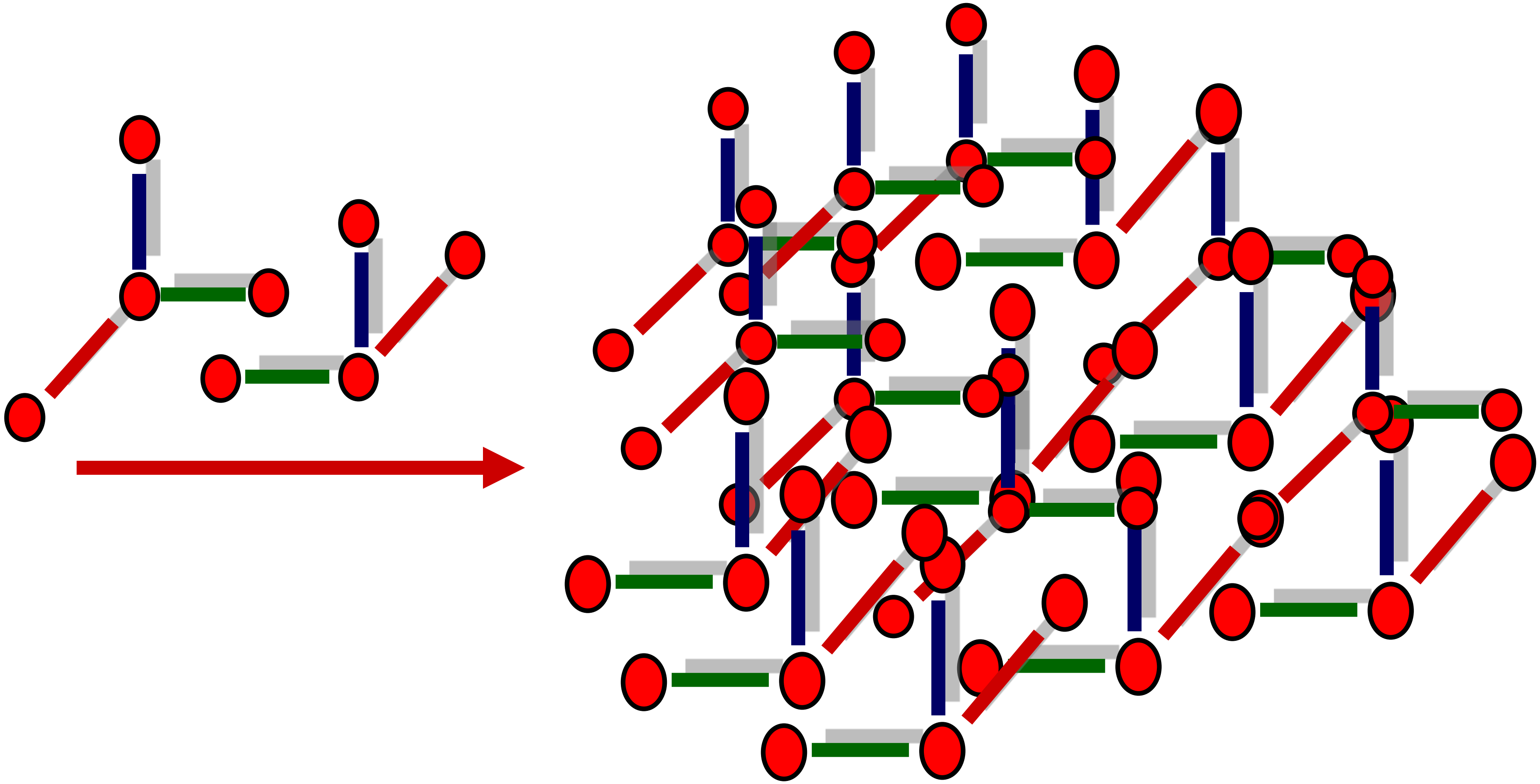




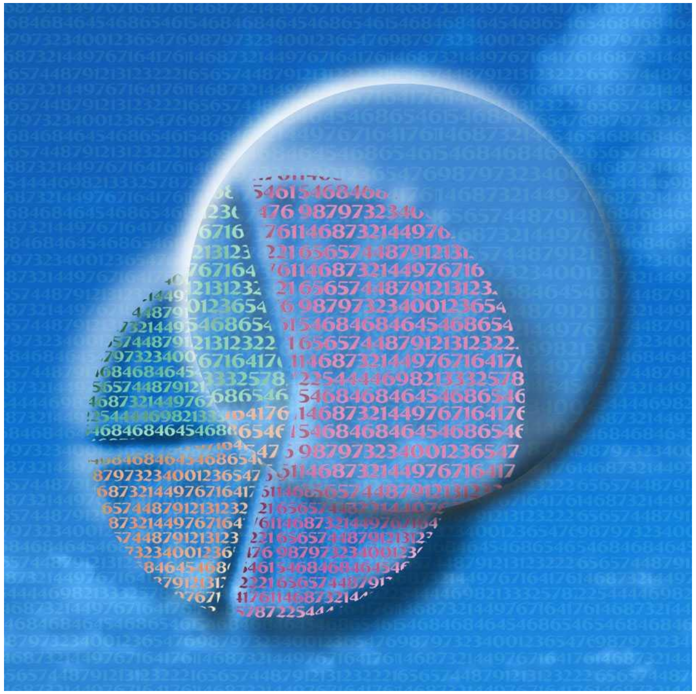


Increasing the number of OAT's the test becomes quantitative…

…because this design is the same used for the total sensitivity index ST



Thus one can start EE-wise (few points) and continue variance-based, without discarding points, by just changing the estimator (from that for Morris to that for ST).



END