Máster Universitario en Administración y Dirección de Empresas
Full Time MBA

Quantitative methods for decision making

Professor Andrea Saltelli

BARCELONA SCHOOL OF MANAGEMENT

Elements of quantification for decision making with emphasis on operation research

BARCELONA SCHOOL OF MANAGEMENT

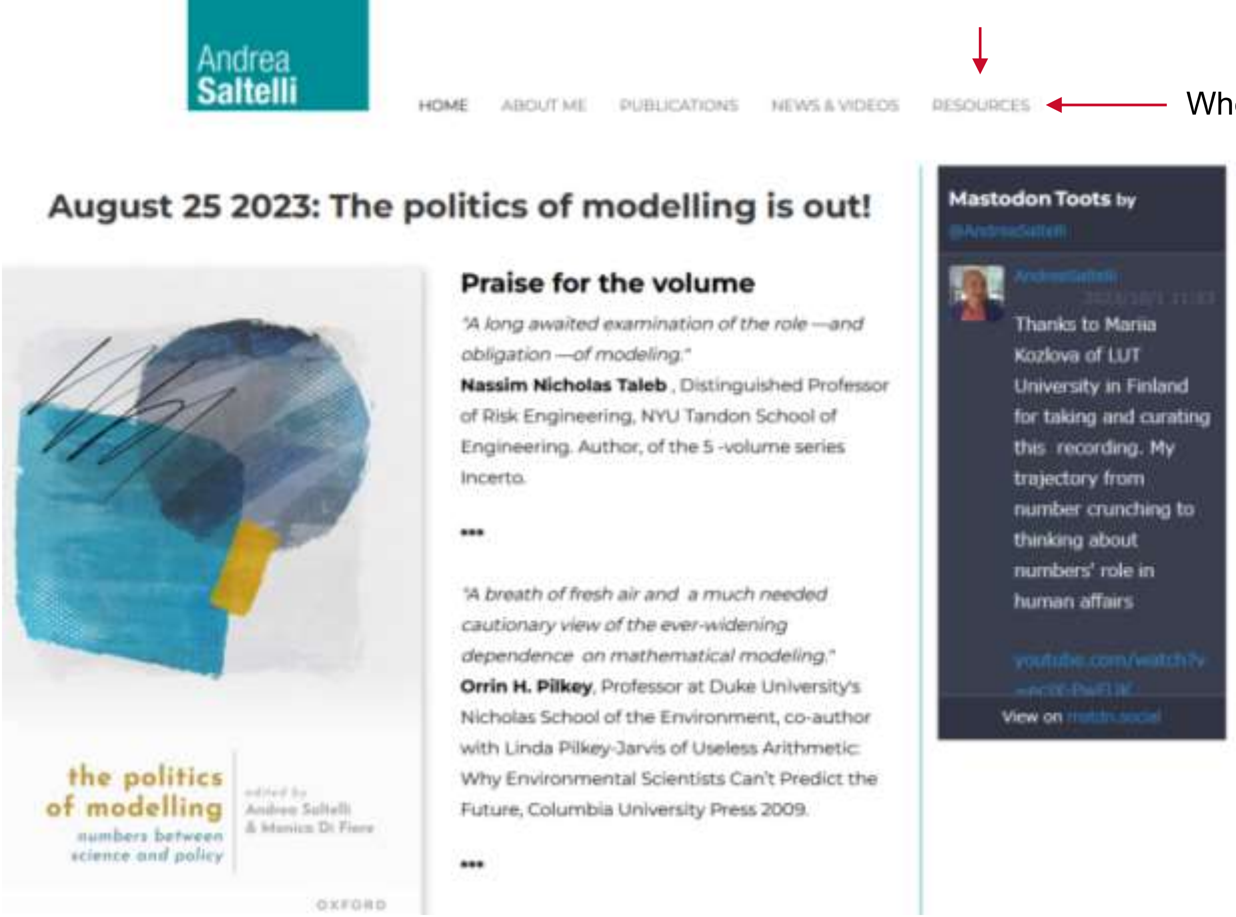
In this set of slides:

16.

Nonlinear Programming

Problem framing and examples. Graphical illustration. Bisection and Newton methods of solution. Metaheuristics. Genetic algorithms. Examples of nonlinear models. Python coding. Exploration versus optimization. Hillier 2014, chapters 13 and 14.

BARCELONA SCHOOL OF MANAGEMENT



Where to find this talk

The talk is also at

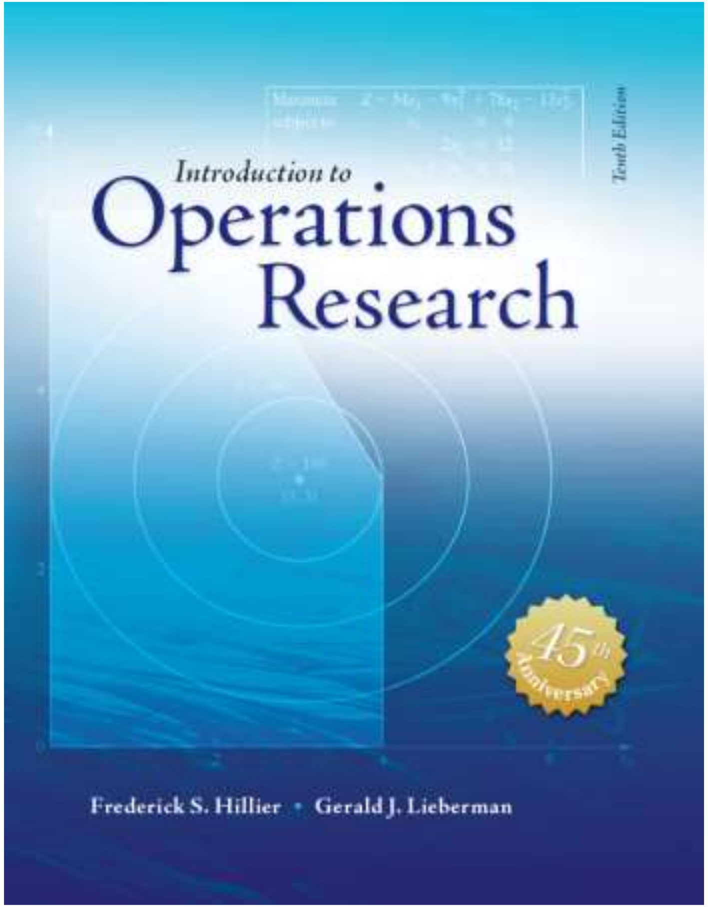
https://ecampus.bsm.upf.edu/,

where you find additional reading material

BARCELONA SCHOOL OF MANAGEMENT

Where to find this book:

https://www.dropbox.com/sh/ddd48a8jguinbcf/AABF0s4eh1lPLVxdx0pes-0fa?dl=0&preview=Introduction+ to+ Operations+ Research+ -+ Frederick+ S.+ Hillier.pdf



upf. BARCELONA SCHOOL OF MANAGEMENT

Problem setting

Finding values of $\boldsymbol{x} = (x_1, x_2, \dots x_n)$ as to maximize or minimize a generic function $f(\boldsymbol{x})$ subject to

$$g_i(\boldsymbol{x}) \leq b_i \quad i = 1,2, \dots m$$

and

$$\boldsymbol{x} \geq 0$$

A Standard Form of the Model:

Maximize $\quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$

Subject to:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m, \end{aligned}$$

And to:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \dots, \quad x_n \geq 0.$$

Z = value of overall measure of performance

x_j = decision variables, level of activity j for $j = 1,2,\dots n$

a_j^i = amount of resource i consumed by each unit of activity j

b_i amount of resource i that is available for allocation to activities $i = 1,2,\dots m$

c_j increase in Z that would result from each unit increase in level of activity

What is the difference from the linear problem?

Problem setting

Finding values of $\boldsymbol{x} = (x_1, x_2, \ldots x_n)$ as to $\boxed{\text{maximize or minimize}}$ a generic function $f(\boldsymbol{x})$ subject to

Is this specification necessary?

$$g_i(\boldsymbol{x}) \leq b_i \quad i = 1,2, \ldots m$$

and

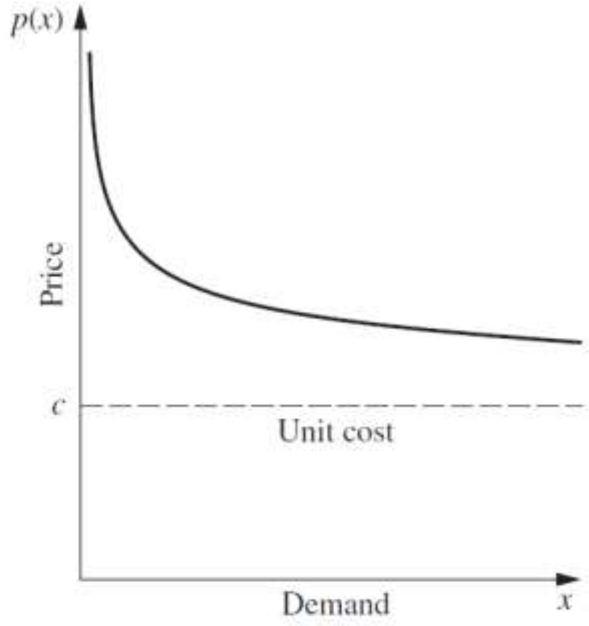
$$\boldsymbol{x} \geq 0$$

upf. BARCELONA SCHOOL OF MANAGEMENT

Many ways in which a linear problem can become nonlinear

Linear: there is a fixed unit profit associated with each product, so the resulting objective function will be linear

Nonlinear: prices $\boldsymbol{p(x)}$ are subject to elasticity



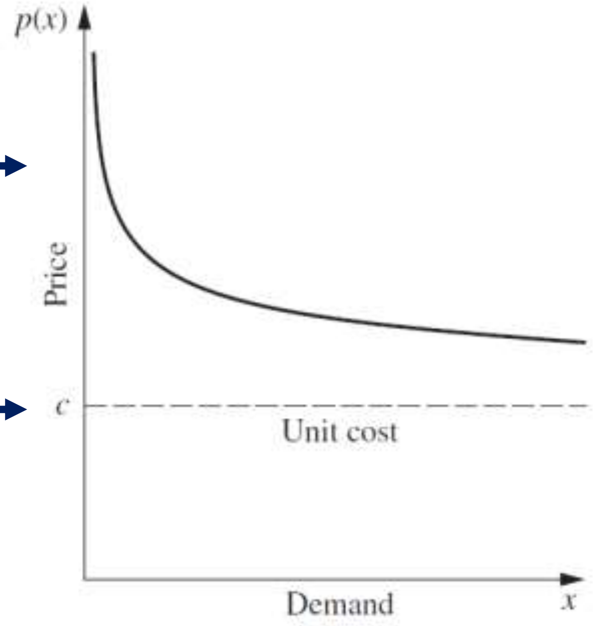
BARCELONA SCHOOL OF MANAGEMENT

The firm's profit P from producing and selling x units is given by the nonlinear function

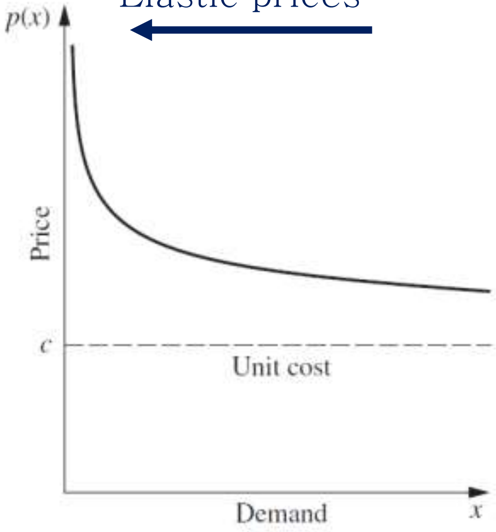
$$P(x) = xp(x) - cx$$

But the higher the price, the less is sold

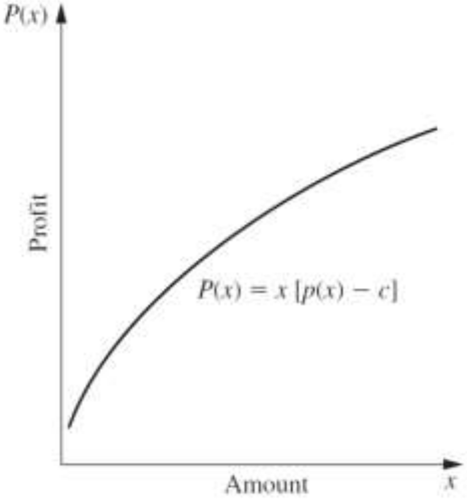
The production cost remain constant



Elastic prices ⋯



⋯ Lead to non linear profits



The firm's profit P from producing and selling x units is given by the nonlinear function

$$P(x) = xp(x) - cx$$

upf. BARCELONA SCHOOL OF MANAGEMENT

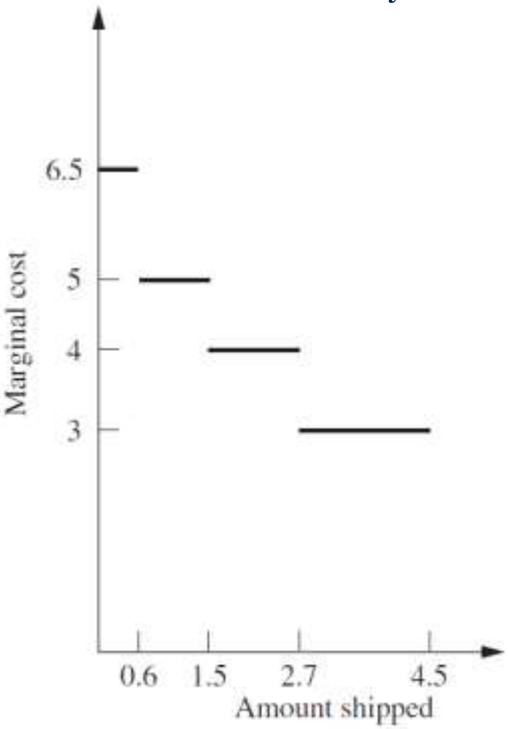
But the production cost may as well be a non linear function, e.g. in the case of increasing or diminishing returns

$$P(x) = xp(x) - cx$$

Source:https://www.bbcgoodfood.com/howto/guide/health-benefits-bananas

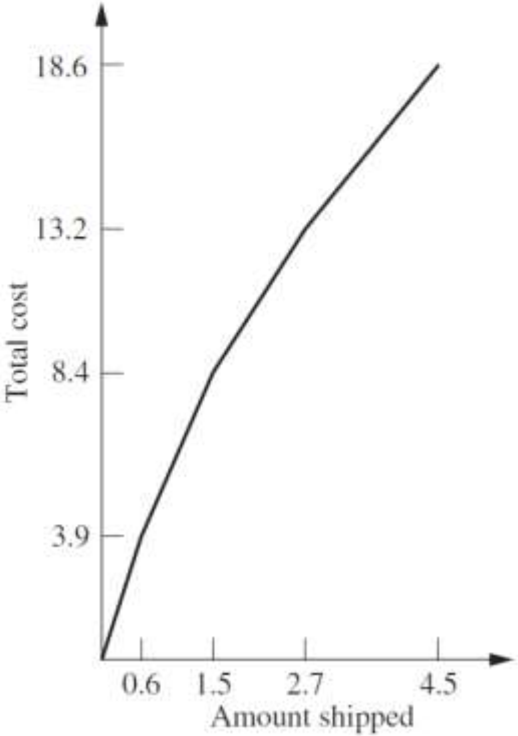
Source:https://www.mercedes-benz.es/passengercars/models/coupe/new/cle.html

In transportation problems the shipping costs may decrease with the volume shipped



Decreasing unitary cost
…

… Piecewise total cost



BARCELONA SCHOOL OF MANAGEMENT

In portfolio modelling the decision variable could be the number of shares of a given stock $x_j, i = 1,2, \dots n$ to be included

Assume known the mean return of stock μ_j and its variance σ_{jj}. Also assume that σ_{jj} is a proxy of the risk for that stock.

But the fluctuations of the stock are not independent, se we also need to know the covariances $\sigma_{ij}, i \neq j$, $i, j = 1,2 \dots n$

Return from the entire portfolio $R(\boldsymbol{x}) = \sum_j^n \mu_j x_j$

Variance $V(\boldsymbol{x})$ of the total return $V(\boldsymbol{x}) = \sum_{j=1}^n \sum_{i=1}^n \sigma_{ij} x_i x_j$

Source: https://www.britannica.com/money/topic/stock-exchange-finance

upf. BARCELONA SCHOOL OF MANAGEMENT

So the nonlinear problem is
Minimize

$$V(\boldsymbol{x}) = \sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_{ij} x_i x_j$$

Subject to

$\sum_j^n \mu_j x_j \geq \boldsymbol{L}$ where $\boldsymbol{L}$ is the minimum profit desired

$\sum_j^n P_j x_j \leq \boldsymbol{B}$ where P_j is the cost of stock j and $\boldsymbol{B}$ is the budget available for the portfolio

$x_j \geq 0, i = 1,2, \dots n$

Source: https://www.britannica.com/money/topic/stock-exchange-finance

upf. BARCELONA SCHOOL OF MANAGEMENT

So the nonlinear problem is
Minimize

$$V(\boldsymbol{x}) = \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} x_i x_j$$

Subject to

$\sum_j^n \mu_j x_j \geq \boldsymbol{L}$ where $\boldsymbol{L}$ is the minimum profit desired

$\sum_j^n P_j x_j \leq \boldsymbol{B}$ where P_j is the cost of stock j and $\boldsymbol{B}$ is the budget available for the portfolio

$x_j \geq 0, i = 1,2, \ldots n$

There is a trade off between $\boldsymbol{L}$ and $V(\boldsymbol{x})$

(higher profit associated to higher risk)

So that this problem is solved for a range of values of $\boldsymbol{L}$ and comparing for each $\boldsymbol{L}$ the associated $R(\boldsymbol{x}) = \sum_j^n \mu_j x_j$ and $V(\boldsymbol{x})$

"Therefore, rather than stopping with one choice of *L*, it is common to use a *parametric* (nonlinear) programming approach to generate the optimal solution as a function of *L* over a wide range of values of *L*. The next step is to examine the values of *R*(**x**) and *V*(**x**) for these solutions that are optimal for some value of *L* and then to choose the solution that seems to give the best trade-off between these two quantities." (Hillier, p. 552)

BARCELONA SCHOOL OF MANAGEMENT

The estimation of the σ_{ij} is a delicate matter; the case of the subprime mortgage crisis

Nassim Nicholas Taleb, hedge fund manager and author of *The Black Swan*, is particularly harsh when it comes to the copula. "People got very excited about the Gaussian copula because of its mathematical elegance, but the thing never worked," he says. "Co-association between securities is not measurable using correlation," because past history can never prepare you for that one day when everything goes south. "Anything that relies on correlation is charlatanism."

« Back to Article

WIRED MAGAZINE: 17.03

Recipe for Disaster: The Formula That Killed Wall Street

By Felix Salmon 02.23.09

upf. BARCELONA SCHOOL OF MANAGEMENT

$$\Pr[T_A<1, T_B<1] = \phi_2(\phi^{-1}(F_A(1)), \phi^{-1}(F_B(1)), \gamma)$$

Here's what killed your 401(k) *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick—and fatally flawed—way to assess risk. A shorter version appears on this month's cover of Wired.*

Here is what killed your 401(k)…

Li's Gaussian copula function …

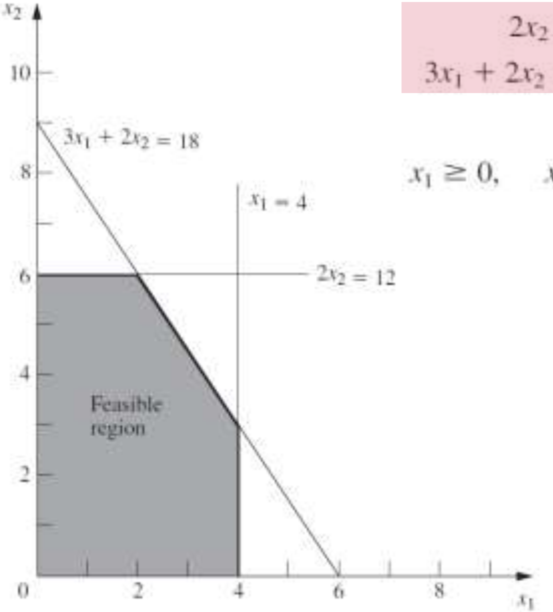
Nassim Nicholas Taleb, hedge fund manager and author of *The Black Swan*, is particularly harsh when it comes to the copula. "People got very excited about the Gaussian copula because of its mathematical elegance, but the thing never worked," he says. "Co-association between securities is not measurable using correlation," because past history can never prepare you for that one day when everything goes south. "Anything that relies on correlation is charlatanism."

Felix Salmon, Wired, February 2009

Source: https://www.wired.com/2009/02/wp-quant/

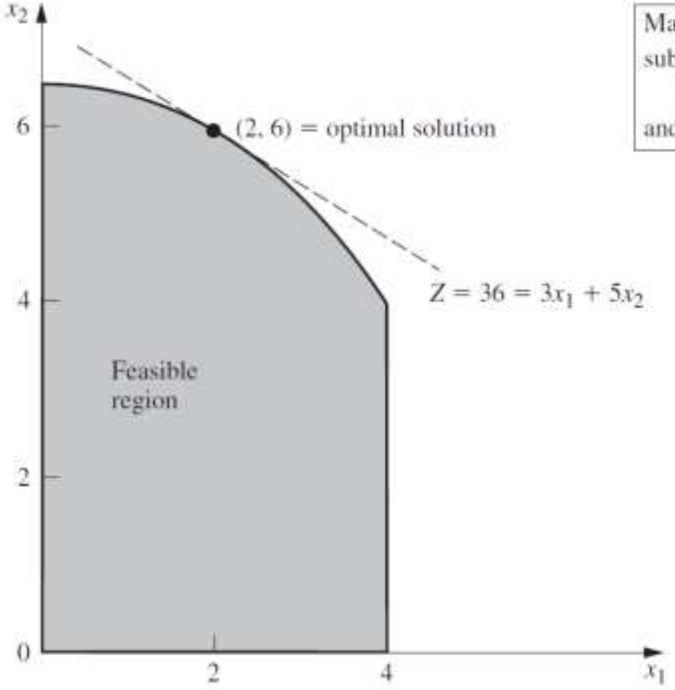
$$\text{Maximize} \quad Z = 3x_1 + 5x_2,$$

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0, \quad x_2 \geq 0.$$

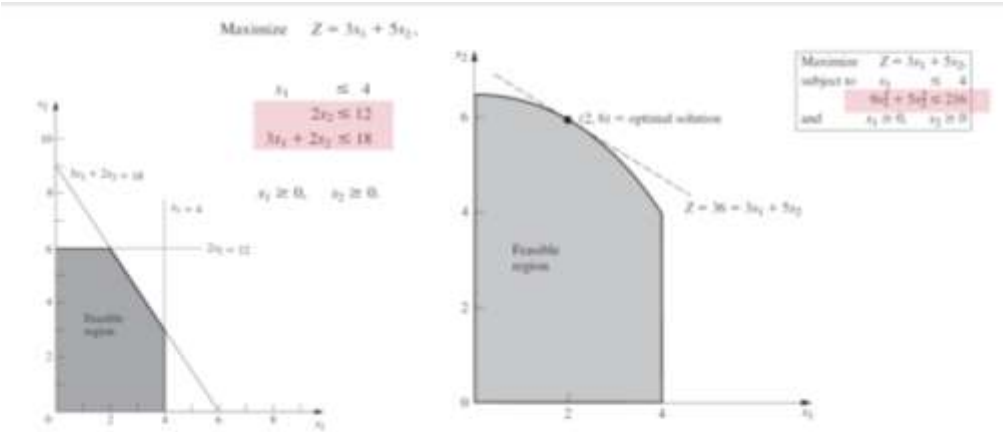


Linear

Maximize	$Z = 3x_1 + 5x_2,$	
subject to	$x_1 \leq 4$	
	$9x_1^2 + 5x_2^2 \leq 216$	
and	$x_1 \geq 0,$	$x_2 \geq 0$



Nonlinear



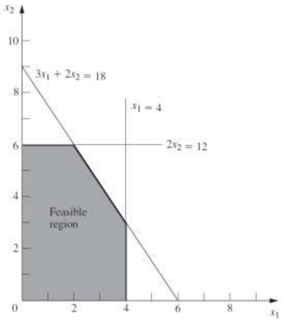
The solution still happens to be on the boundary of the feasible region, but there are no longer the corner points feasible (CPF) to help us

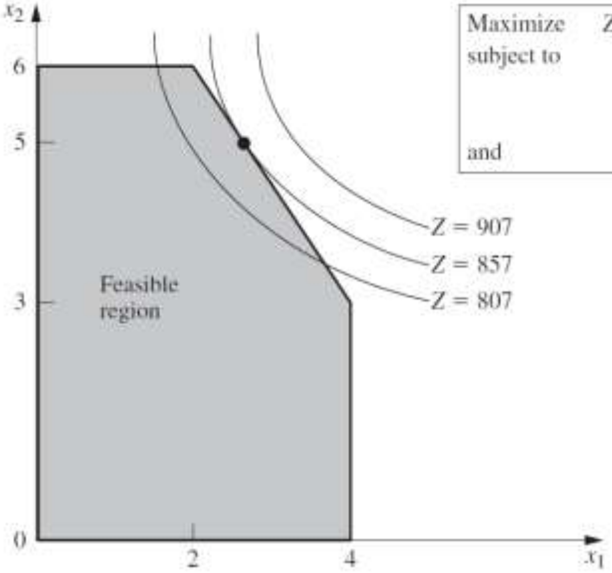
If we now maintain the old constraints of the linear problem i.e.

$$
\begin{aligned}
x_1 &\le 4 \\
2x_2 &\le 12 \\
3x_1 + 2x_2 &\le 18 \\
x_1 &\ge 0 \\
x_2 &\ge 0
\end{aligned}
$$

But change the objective function to a nonlinear form

$$Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2$$



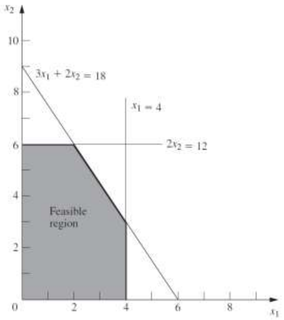


Maximize	$Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2,$	
subject to	$x_1 \le 4$	
	$2x_2 \le 12$	
	$3x_1 + 2x_2 \le 18$	
and	$x_1 \ge 0,\quad x_2 \ge 0$	

BARCELONA SCHOOL OF MANAGEMENT

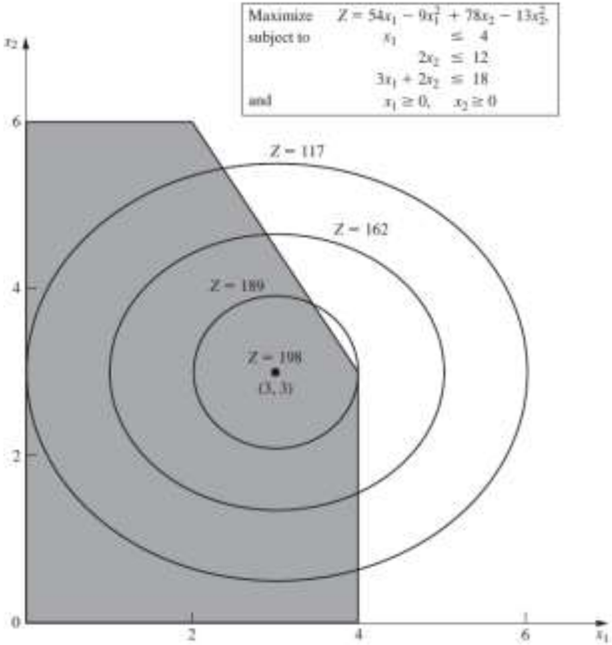
If we now maintain the old constraints of the linear problem i.e.

$$
\begin{aligned}
x_1 &\leq 4 \\
2x_2 &\leq 12 \\
3x_1 + 2x_2 &\leq 18 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
$$

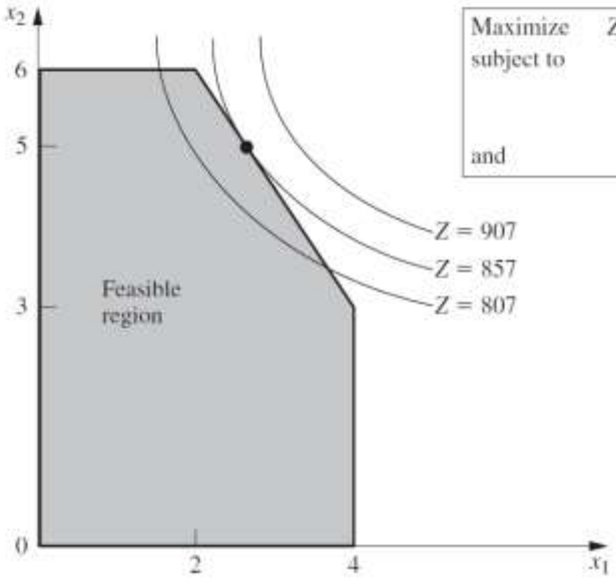


…or to another nonlinear form

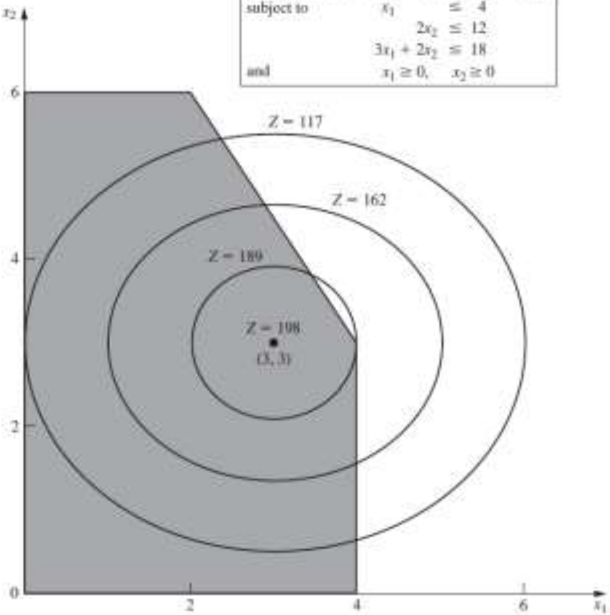
$$Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2$$



$$
\begin{aligned}
\text{Maximize} \quad & Z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2, \\
\text{subject to} \quad & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
\text{and} \quad & x_1 \geq 0, \quad x_2 \geq 0
\end{aligned}
$$



$$
\begin{aligned}
\text{Maximize} \quad & Z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2, \\
\text{subject to} \quad & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
\text{and} \quad & x_1 \geq 0, \quad x_2 \geq 0
\end{aligned}
$$



The solution can be anywhere in the feasible region, no longer just on its frontier

Other complications of the nonlinear problem: there can be more maxima

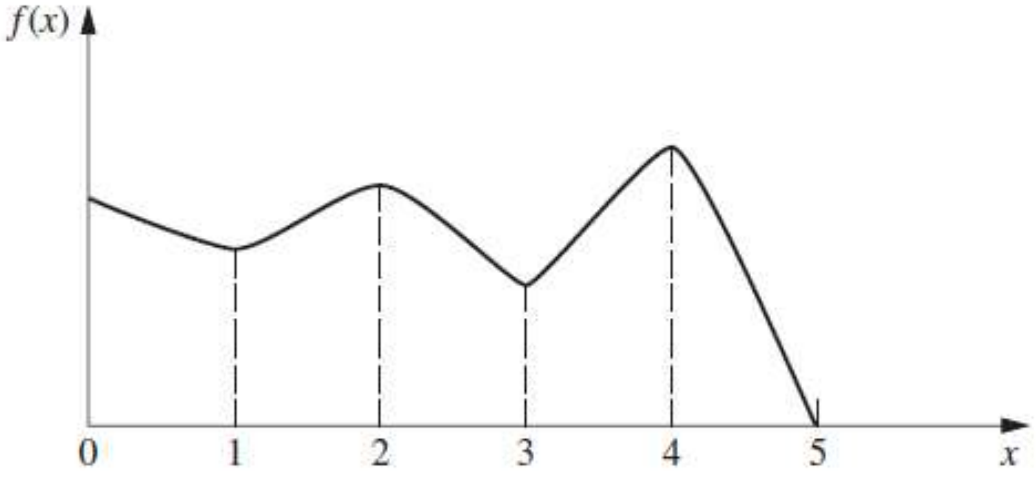


FIGURE 13.8
A function with several local maxima ($x = 0, 2, 4$), but only $x = 4$ is a global maximum.

In order to have just one maximum the objective function must be **concave** over the dominion of the search. Similarly for a **convex** function there is guarantee of just one minimum

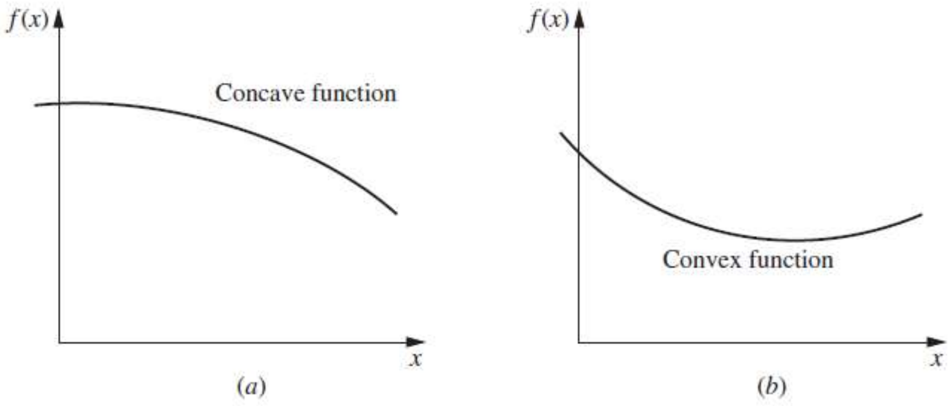


FIGURE 13.9
Examples of (*a*) a concave function and (*b*) a convex function.

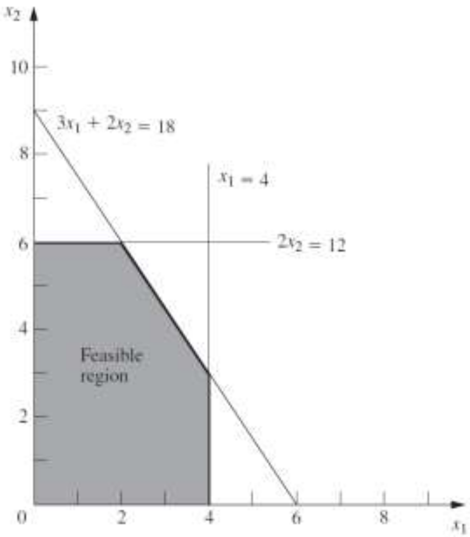
$$\frac{\partial^2 f(x)}{\partial x^2} \leq 0 \qquad \frac{\partial^2 f(x)}{\partial x^2} \geq 0$$

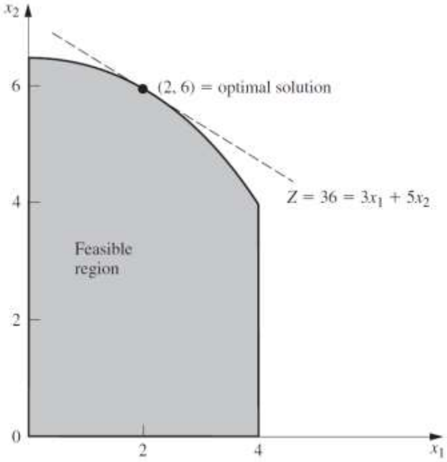
No constraints here!

In the case of constraints the existence of a global maximum is conditioned by the shape of the feasible region. If the feasible region is a **convex set**, then the global maximum is ensured.

"A convex set is simply a set of points such that, for each pair of points in the collection, the entire line segment joining these two points is also in the collection"

Note: in linear programming the feasible region is always a convex set



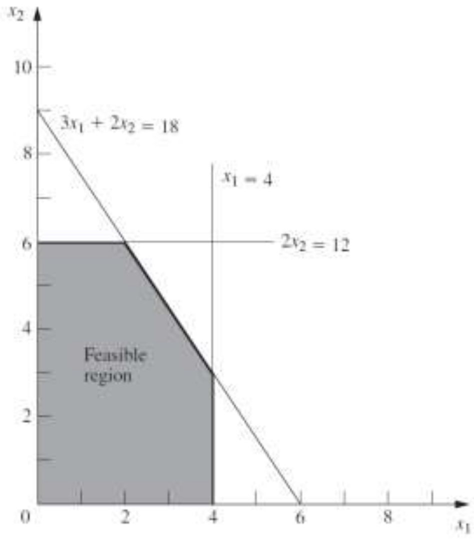


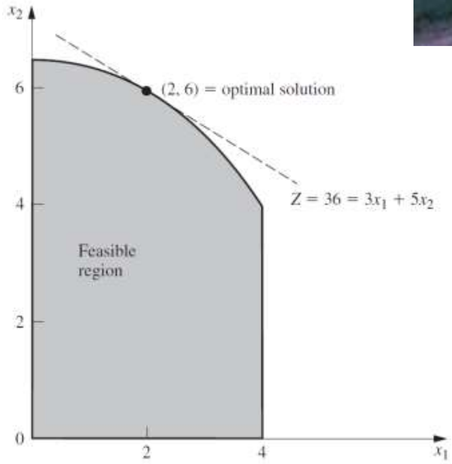
The feasible region is a convex set in both these cases

BARCELONA SCHOOL OF MANAGEMENT

= you can reach each point in the set from each other point in the set by walking in a straight line

Source: https://mwpetersonlaw.com/

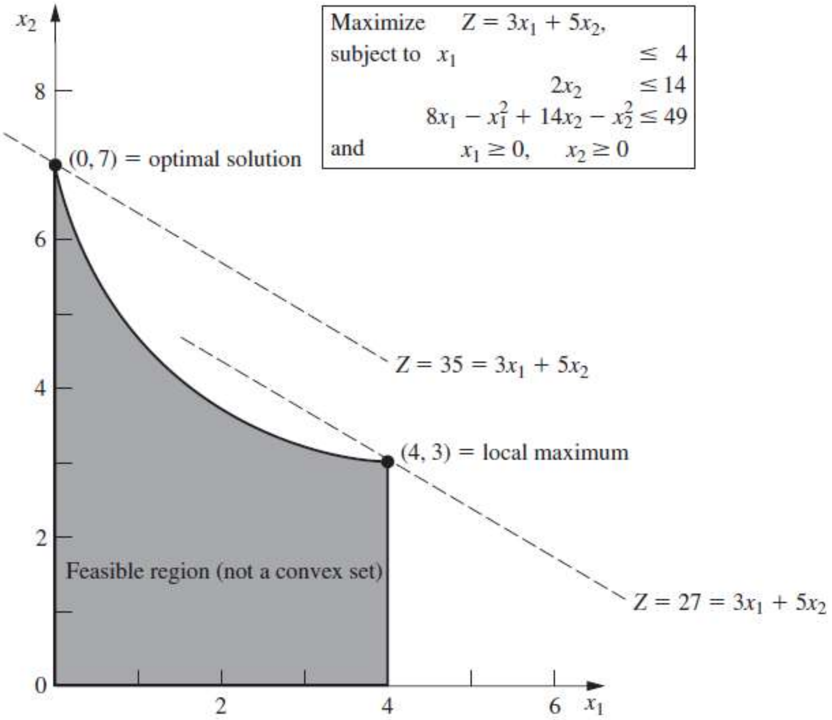




The feasible region is a convex set in both these cases

BARCELONA SCHOOL OF MANAGEMENT

This is not the case here



A nonlinear programming problem with constraints $g_i(\boldsymbol{x}) \leq b_i \quad i = 1,2, \ldots m$ has a **global maximum** when the objective $f(x)$ is a concave function and all the $g_i(\boldsymbol{x})$ are convex

➔ This is known as convex programming problem

BARCELONA SCHOOL OF MANAGEMENT

The minimum can be anywhere;

Exercise:

- unconstrained problem, write a function $f(x)$ whose maximum is in $(x = 0)$
- unconstrained problem, write a function $f(x_1, x_2)$ whose maximum is in $(3,3)$
- Find the maximum of $f(x) = 6x - x^2$
- Draw $f(x) = 6x - x^2$

The minimum/maximum can be anywhere;

Exercise:

- unconstrained problem, write a function $f(x)$ whose maximum is in $(x = 0)$
  - $f(x) = -x^2$
- unconstrained problem, write a function $f(x_1, x_2)$ whose minimum is in $(3,3)$
  - $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$
- Find the maximum of $f(x) = 6x - x^2$
  - There will be a maximum in $x = 3$ (why?)
- Draw $f(x) = 6x - x^2$

Last lesson: we were here

AN

Please write a program in Python to plot 6x-x**2

To plot the function `6x - x**2` in Python, you can use the `matplotlib` library. If you haven't already installed `matplotlib`, you can do so using `pip`:

bash Copy code

```bash
pip install matplotlib
```

Once you have `matplotlib` installed, you can create a simple Python script to generate the plot. Here's an example program:

python Copy code

```python
import numpy as np
import matplotlib.pyplot as plt

# Define the range of x values
x = np.linspace(0, 6, 100)  # Generate 100 evenly spaced points between 0 an
```

Ask ChatGTP

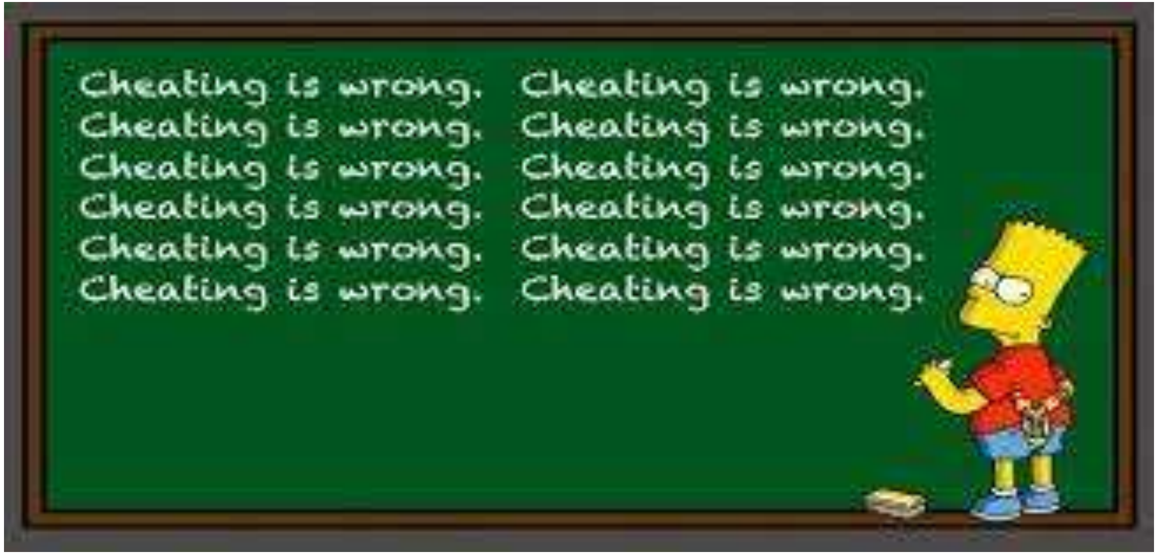
Go to

https://openai.com/

Login or create account

BARCELONA SCHOOL OF MANAGEMENT

Is using ChatGPT cheating?



Source: The Simpsons, Twentieth Century Fox Film Corporation

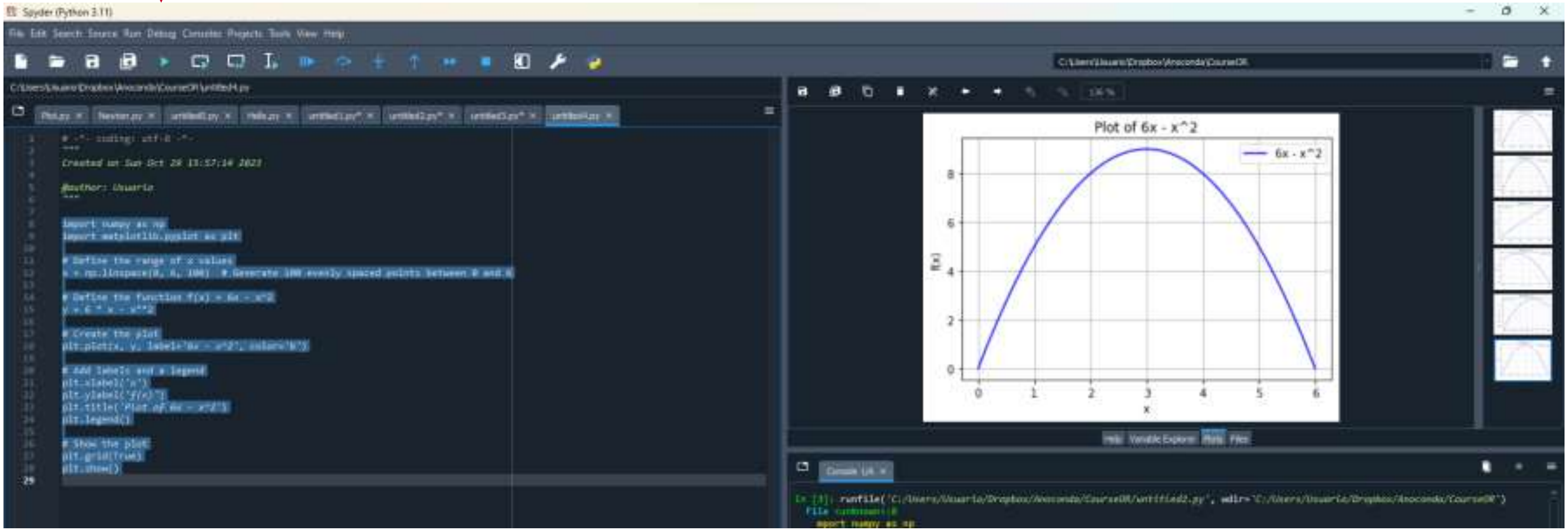
upf. BARCELONA SCHOOL OF MANAGEMENT

python Copy code

```python
import matplotlib.pyplot as plt
import numpy as np

# Define the x values
x = np.linspace(0, 6, 100)  # Generate 100 evenly spaced points between 0 an

# Calculate the corresponding y values using the function 6x - x^2
y = 6 * x - x**2

# Create a plot
plt.plot(x, y, label='6x - x^2')

# Add labels and a legend
plt.xlabel('x')
plt.ylabel('y')
plt.title('Plot of 6x - x^2')
plt.legend()

# Show the plot
plt.grid(True)
plt.show()
```

Launch Anaconda,
launch Spyder,
execute

Execute

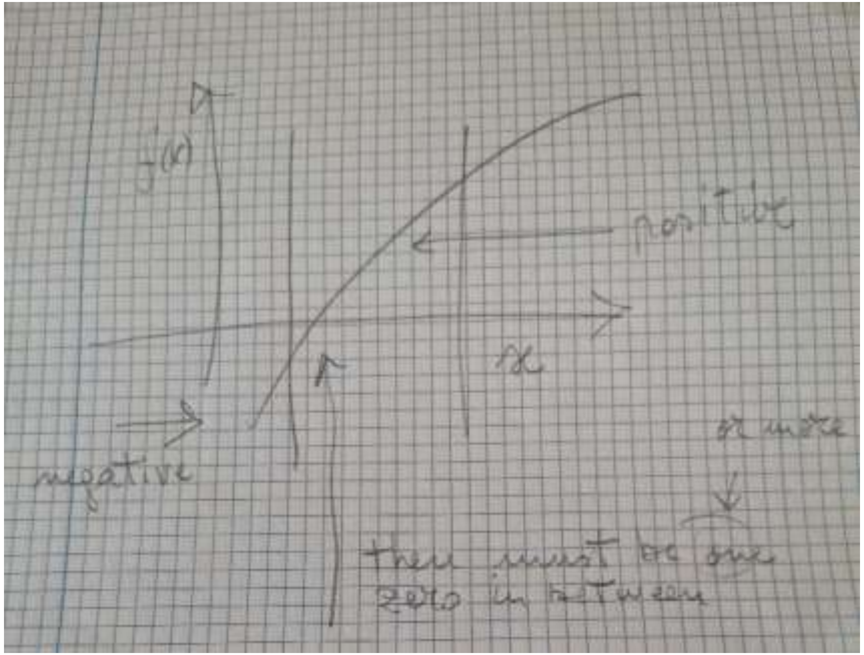


Spyder (Python 3.11)
File Edit Search Source Run Debug Consoles Projects Tools View Help
Plot of 6x - x^2
6x - x^2
f(x)
x

Bisection method (Bolzano search plan) – for concave functions.

If a continuous function has values of opposite sign inside an interval, then it has a root in that interval (Bolzano's theorem)

Not rocket science!



Bernardus Placidus Johann Nepomuk Bolzano, 1781–1848
Source: Wikipedia Commons

upf. BARCELONA SCHOOL OF MANAGEMENT

Bisection method (Bolzano search plan) – for concave functions.

If a continuous function has values of opposite sign inside an interval, then it has a root in that interval (Bolzano's theorem)

Applying this theorem to the derivative $\frac{\partial f(x)}{\partial x}$, knowing that for concave functions the maximum corresponds to the point where $\frac{\partial f(x)}{\partial x} = 0$ one can first identify an interval where $\frac{\partial f(x)}{\partial x}$ changes sign, then reduce the dimension interval iteratively to get the solution

Bernardus Placidus Johann Nepomuk Bolzano, 1781–1848 Source: Wikipedia Commons

BARCELONA SCHOOL OF MANAGEMENT

Notation

x^* solution being sought

ε tolerance in the search of x^*

x' current trial solution

x_l current lower bound

x_u current upper bound

We want the maximum of

$$f(x) = 6x - x^2 - \frac{x^3}{3}$$

Setting the tolerance ε at 0.1 (10%)

UPF Barcelona School of Management

Procedure: Find extreme x_u and x_l so that $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_l}$<0, while $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_u}$>0 and initialize $x' = \frac{x_l+x_u}{2}$

1) Evaluate $\frac{\partial f(x)}{\partial x}$ at $\mathrm{x} = x'$

2) if $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x\prime}$<0 redefine $x_l = x'$

3) if $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x\prime}$ >0 redefine $x_u = x'$

4) Update $x' = \frac{x_l+x_u}{2}$

Iterate 1–4 till $x_u - x_l \leq 2\varepsilon$ so that new x'must be within ε of x^*

BARCELONA SCHOOL OF MANAGEMENT

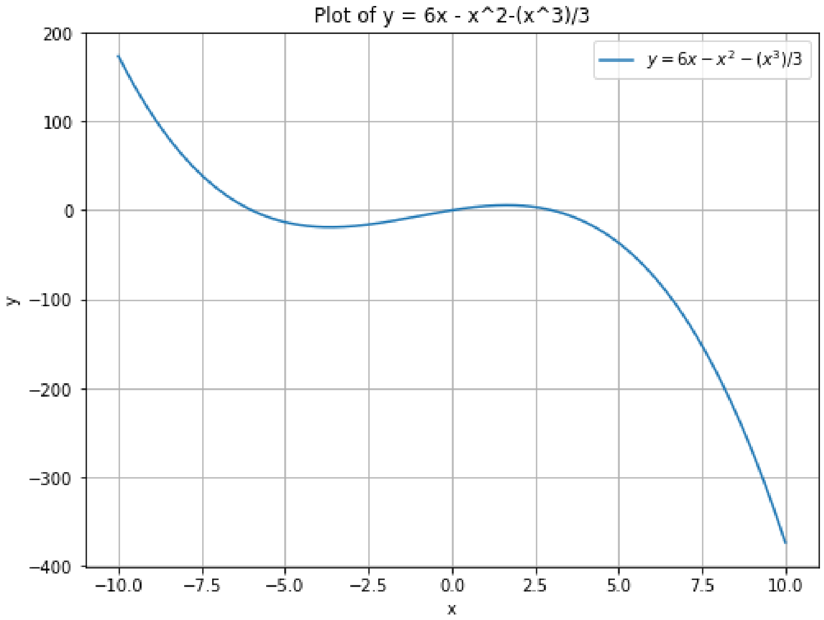
Use the procedure to find a local maximum of

$$f(x) = 6x - x^2 - \frac{x^3}{3}$$

Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

Between 1. and 2. Set the tolerance ε at 0.1 (10%)

Help: $\frac{\partial f(x)}{\partial x} = 6 - 2x - x^2$



Use the procedure to find in interval (1.,2.) the maximum of

$$f(x) = 6x - x^2 - \frac{x^3}{3}$$
$$\frac{\partial f(x)}{\partial x} = 6 - 2x - x^2$$

Set the tolerance ε at 0.1 (10%)

By hand, I get x=1.72, $\frac{\partial f(x)}{\partial x}$ ~0.4 after 3 iterations and some number crunching; search interval $x \in (1,2)$

The procedure is laborious (slow convergence)

Procedure

Find extreme x_u and x_l so that $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_l}$<0, while $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_u}$>0 and initialize $x' = \frac{x_l+x_u}{2}$

1) Evaluate $\frac{\partial f(x)}{\partial x}$ at x = x'

2) if $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x'}$<0 redefine $x_l = x'$

3) if $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x'}$>0 redefine $x_u = x'$

4) Update $x' = \frac{x_l+x_u}{2}$

Iterate 1–4 till $x_u - x_l \leq 2ε$ so that new x'must be within ε of x^*

BARCELONA SCHOOL OF MANAGEMENT

Newton's method – for concave functions

Newton is credited to have discovered calculus (in parallel with Leibniz) and his method is that of a quadratic approximation based on a truncated Taylor series

Isac Newton (1643-1727)
Source: Wikipedia Commons

$$f(x_{i+1}) = f(x_i) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i} (x_{i+1} - x_i) + \frac{1}{2}\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i} \left((x_{i+1} - x_i)\right)^2 + \frac{1}{6}\left.\frac{\partial^3 f(x)}{\partial x^3}\right|_{x=x_i} \left((x_{i+1} - x_i)\right)^3 \cdots + \frac{1}{n!}\left.\frac{\partial^n f(x)}{\partial x^n}\right|_{x=x_i} \left((x_{i+1} - x_i)\right)^n$$

We stop (truncate) now at the second order term

upf. BARCELONA SCHOOL OF MANAGEMENT

Looking at this as a function of only x_{i+1}, with x_i and its derivative as fixed

$$f(x_{i+1}) = f(x_i) + \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i} (x_{i+1} - x_i) + \frac{1}{2}\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i} (x_{i+1} - x_i)^2 + \cdots$$

We can differentiate with respect to x_{i+1} to get

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_{i+1}} = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i} + \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i} (x_{i+1} - x_i)$$

Setting this to zero (as to find the maximum) and reordering we get

$$x_{i+1} = x_i - \frac{\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}}{\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}}$$

Source: The Simpson, 20th Television Animation (The Walt Disney Company)

Using this beauty $x_{i+1} = x_i - \frac{\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}}{\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}}$

The procedure is simple: find a trial value x_i by inspection

1) Evaluate $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}$

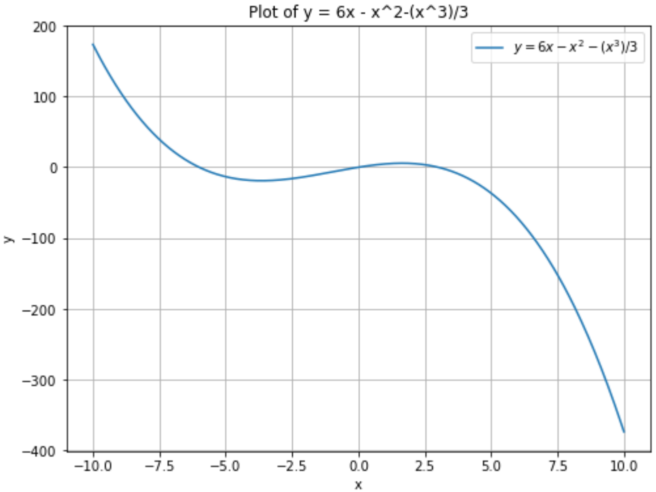
2) Evaluate $\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}$

3) Compute x_{i+1} as from the formula above

Iterate 1–3 till $|x_{i+1} - x_i| \leq \varepsilon$

Exercise: Compute at least up to the first iteration, starting from $x_i = 1$ (**at home, facultative**)

$$f(x) = 6x - x^2 - \frac{x^3}{3}; \qquad \frac{\partial f(x)}{\partial x} = 6 - 2x - x^2; \qquad \frac{\partial^2 f(x)}{\partial x^2} = -2 - 2x$$



Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

BARCELONA SCHOOL OF MANAGEMENT

Using this beauty $x_{i+1} = x_i - \dfrac{\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}}{\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}}$

The procedure is simple: find a trial value x_i by inspection

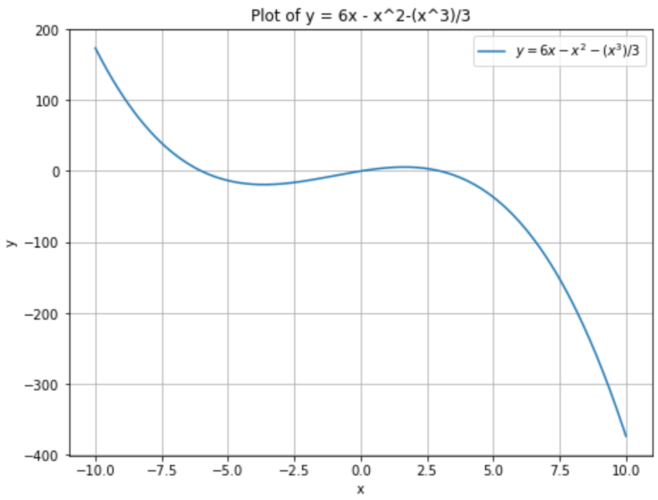
1) Evaluate $\left.\frac{\partial f(x)}{\partial x}\right|_{x=x_i}$

2) Evaluate $\left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x=x_i}$

3) Compute x_{i+1} as from the formula above

Iterate 1–3 till $|x_{i+1} - x_i| \leq \varepsilon$, starting from $x_i = 1$

Phyton code can be used again



BARCELONA SCHOOL OF MANAGEMENT

Spyder (Python 3.11)

File Edit Search Source Run Debug Consoles Projects Tools View Help

C:\Users\Usuario\Dropbox\Anaconda\CourseOR\untitled0.py

temp.py* × untitled0.py* ×

```python
# -*- coding: utf-8 -*-
"""
Created on Tue Aug 29 11:58:16 2023

@author: Usuario
"""

def func(x):
    return 6*x - x**2 - (x**3)/3

def first_derivative(x):
    return 6 - 2*x - x**2

def second_derivative(x):
    return -2 - 2*x

def newton_max_search(starting_point, tolerance=1e-6, max_iterations=100):
    x = starting_point

    for i in range(max_iterations):
        prev_x = x
        x = x - first_derivative(x) / second_derivative(x)

        print(f"Iteration {i+1}: x = {x:.6f}, y = {func(x):.6f}")

        if abs(x - prev_x) < tolerance:
            print("\nConverged!")
            return x

    print("\nMaximum not found within the specified number of iterations.")
    return None

starting_point = 1
max_value = newton_max_search(starting_point)

if max_value is not None:
    print(f"\nMaximum value: y = {func(max_value):.6f} at x = {max_value:.6f}")
```

Question: Please find the maximum of the function y=6x−x**2-(x**3)/3, using the method of Newton pure, using the first and the second derivative, with starting point x=1, and printing out all steps of the search.

To find the maximum of the function $y = 6x - x^2 - \frac{x^3}{3}$ using the Newton-Raphson method with the first and second derivatives, you can follow these steps:

1. Define the function $y = 6x - x^2 - \frac{x^3}{3}$.
2. Calculate the first and second derivatives of the function.
3. Implement the Newton-Raphson method using the formula: $x_{\text{new}} = x_{\text{old}} - \frac{f'(x_{\text{old}})}{f''(x_{\text{old}})}$.
4. Iterate until the change in x is smaller than a certain tolerance or a maximum number of iterations is reached.
5. Print out each iteration's x value and the corresponding function value.

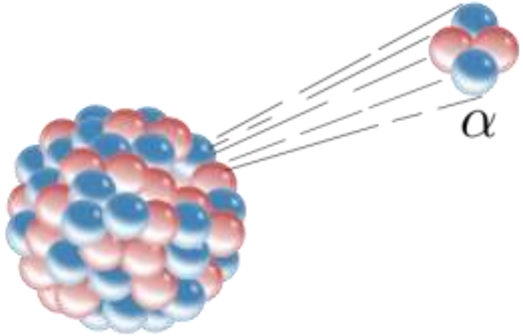
Here's how you can implement this in Python:

```
Iteration 1: x = 1.750000, y = 5.651042
Iteration 2: x = 1.647727, y = 5.680162
Iteration 3: x = 1.645752, y = 5.680173
Iteration 4: x = 1.645751, y = 5.680173
```

upf. BARCELONA SCHOOL OF MANAGEMENT

A different model, where we explore rather than optimize

The Bateman equations, initially written for a radioactive decay chain, can be used to describe any process where transfer occur among compartments without backward term



Source: Wikipedia common

$\partial N_1 / \partial t = -\lambda_1 N_1$ N_1 concentration of species 1, disappear proportionally to itself …

$\partial N_2 / \partial t = \lambda_1 N_1 - \lambda_2 N_2$ …to become N_2 that in turn disappear proportionally to itself…

… …and so on…

$\partial N_n / \partial t = \lambda_{n-1} N_{n-1} - \lambda_n N_n$ …till and end−product is reached

We choose this as it is nonlinear and the differential equation has a neat analytic solution

$\partial N_1 / \partial t = -\lambda_1 N_1$

$\partial N_2 / \partial t = \lambda_1 N_1 - \lambda_2 N_2$

…

$\partial N_n / \partial t = \lambda_{n-1} N_{n-1} - \lambda_n N_n$

For $N_1(0) \neq 0$, $N_i(0) = 0 \ \forall i \neq j$ the solution is

$$N_k(t) = \frac{N_1(0)}{\lambda_k} \sum_{i=1}^{k} \lambda_i \alpha_i \, e^{-\lambda_i t}$$

With

$$\alpha_i \prod_{j=1, j \neq i}^{k} \frac{\lambda_i}{\lambda_j - \lambda_i}$$

Interested in playing with this function in Python? Script in eCampus

A last model, where we revisit our simple linear form (from Lesson 3): $y = \sum_{i=1}^{k} \Omega_i Z_i$

Where y (a scalar) is the output of interest, the Ω_i's **were** fixed coefficients and Z_i's are uncertain input factors following a Normal distribution

$Z_i \sim N\left(\overline{z_i}, \sigma_{Z_i}\right)$

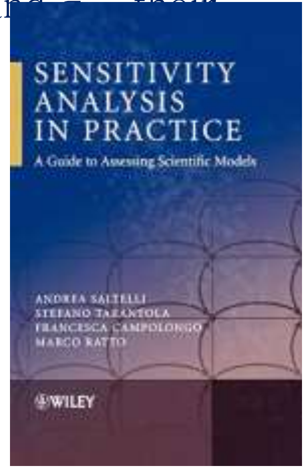
Where $\overline{z_i} = 0, i = 1,2, \ldots k$ are the means of the factors Z_i's and σ_i their standard deviations

We now allow the Ω_i to **be uncertain as well**

$\Omega_i \sim N\left(\overline{\Omega_i}, \sigma_{\Omega_i}\right)$ where $\overline{\Omega_i} = 0, i = 1,2, \ldots k$ are the means of the factors Ω_i's and σ_{Ω_i} their standard deviations

Interested in playing with this function in Python? Script in eCampus

In this book we took $\overline{z_i}$ to be zero, and called $y = \sum_{i=1}^{k} \Omega_i Z_i$ a balanced portfolio, where the Z_i are the assets and the Ω_i the amount held of each security



upf. BARCELONA SCHOOL OF MANAGEMENT

There are as well stochastic search method (called meta-heuristics in the Hillier's book) where the search is done iteratively with trial points and rules to point the search in the right direction, without being greedy

- Tabu Search (don't go there if you have been there already)
- Simulated Annealing (you can walk in the wrong direction but with lower probability)
- Genetic Algorithms (let the fitter reproduce themselves)

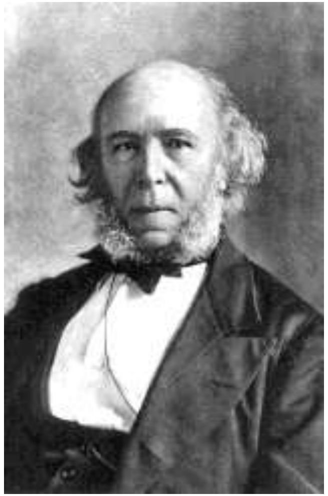
↑

We study this

Genetic algorithms (let the **fitter** reproduce themselves)

Note: Darwin used ‘fittest’, borrowing the term from Spencer, but ‘fitter’ is more apt to his theory as well as to what genetic algorithms do.

“Darwin did not consider the process of evolution as the survival of the fittest; he regarded it as survival of the fitter, because the “struggle for existence” is relative and thus not absolute. Instead, the winners with respect to species within ecosystems could become losers with a change of circumstances” (https://www.britannica.com)

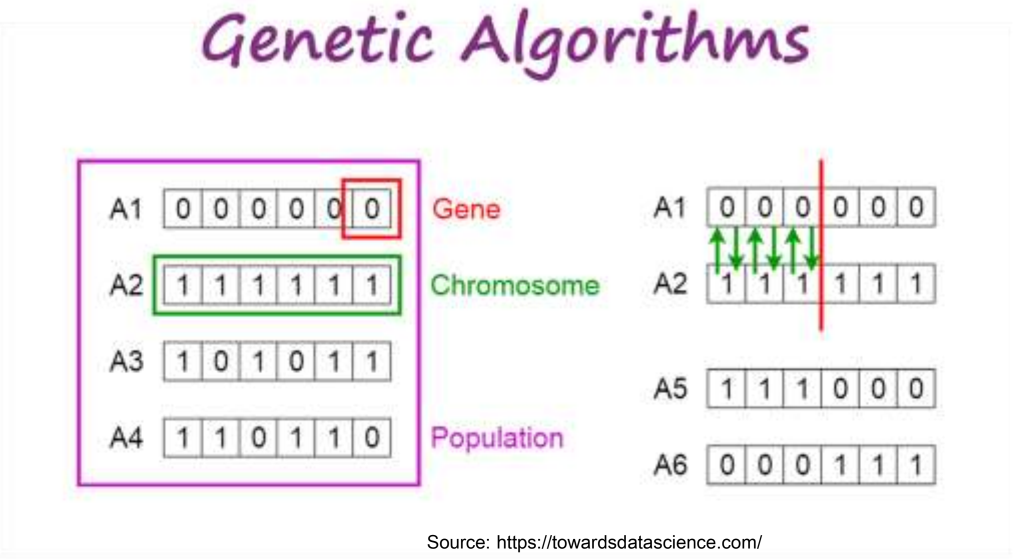


Herbert Spencer (1820–1903)

Charles Darwin (1809–1882)

upf. BARCELONA SCHOOL OF MANAGEMENT

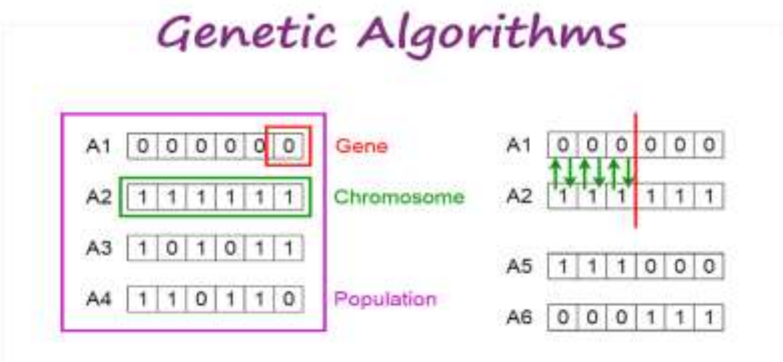
Unlike Tabu search and Simulated Annealing, genetic algorithms do not work with a wandering point, but with an evolving population – a collection of candidate points is generated right at the start, then these generate offspring



Parents

Children

At each generation the parents with the higher fitness have higher probability of reproducing, with each parent passing part of his genes to the offspring ➔ fitness in terms of objective function Z



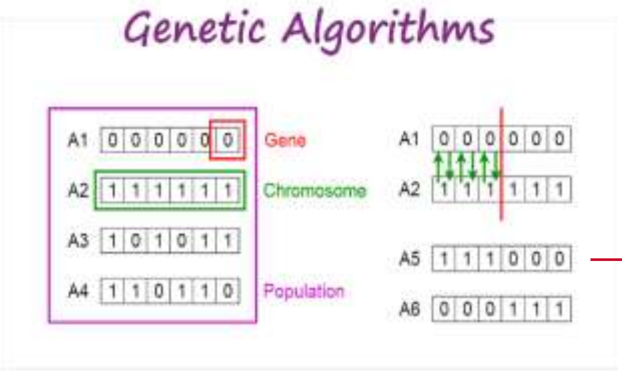
Parents

Children

Source: https://towardsdatascience.com/

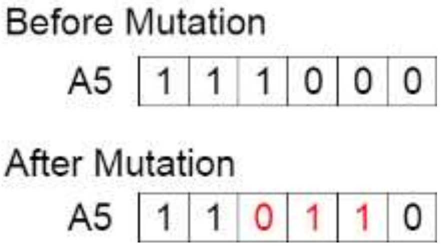
BARCELONA SCHOOL OF MANAGEMENT

Additional random mutations can occur ('errors' in the transcription of DNA or epigenetic factors in the genetic metaphor)



Parents

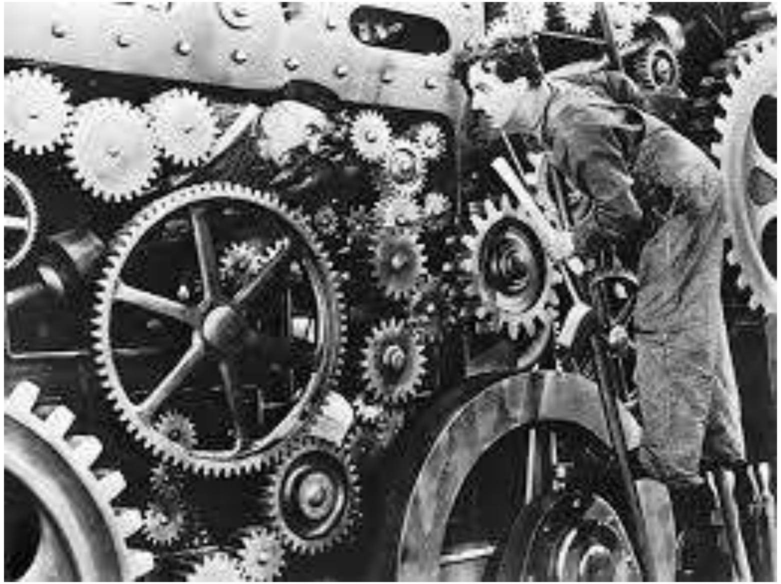
Children



Source: https://towardsdatascience.com/

Genetic algorithms have lots of movable parts! In order to use them you must decide

- Size and composition of the initial population
- How to select the parents based on Z
- How to exchange the genes of the parent to generate the children
- Mutation rate
- Stopping rule

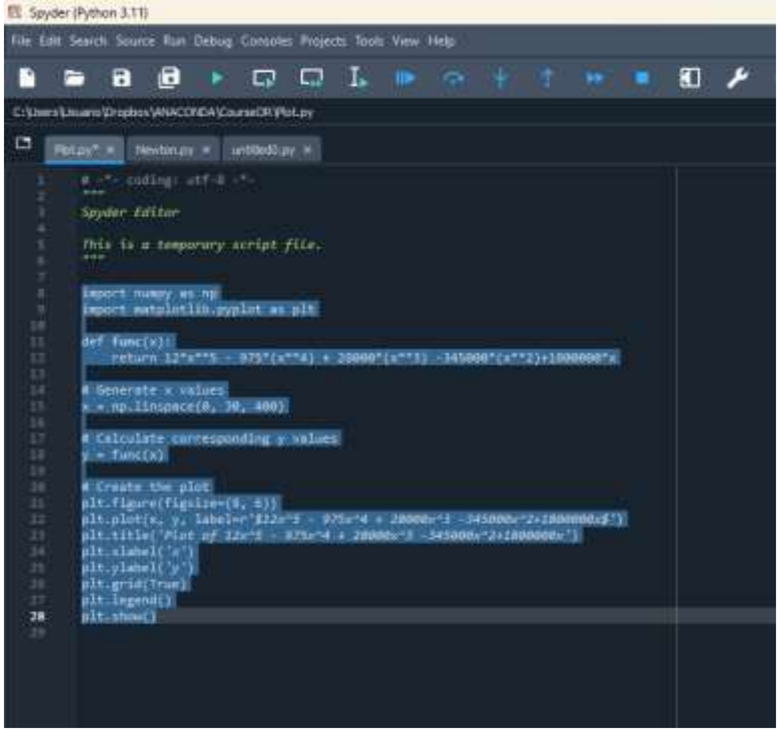


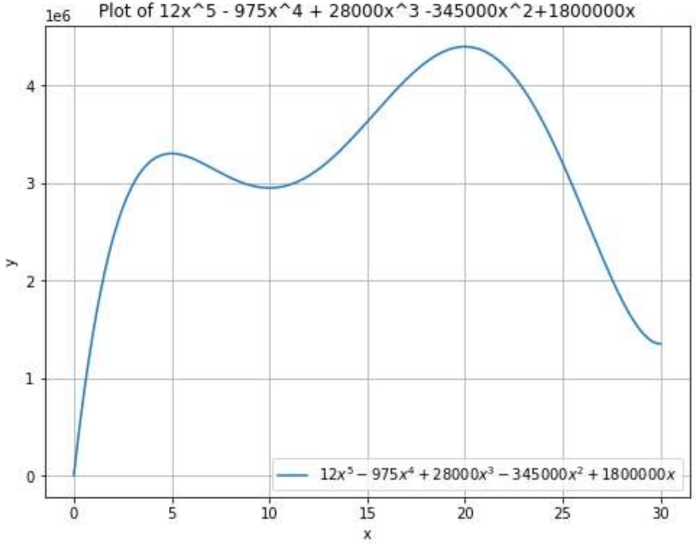
Source: Charlie Chaplin's Modern Times

BARCELONA SCHOOL OF MANAGEMENT

Try with GA with the monster (Hillier, Chapter #14)

$$y = 12x^5 - 975x^4 + 28{,}000x^3 - 345{,}000x^2 + 1{,}800{,}000x^5$$





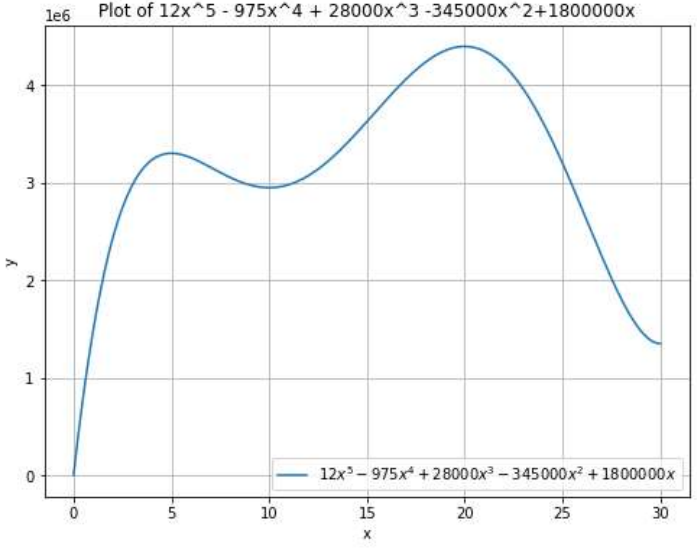
We trick the monster by adding the constraint the x must be integer

BARCELONA SCHOOL OF MANAGEMENT

Try with GA with the monster (Hillier, Chapter #14)
$y = 12x^5 - 975x^4 + 28{,}000x^3 - 345{,}000x^2 + 1{,}800{,}000x^5$

Since the x axis spans from zero to 32, we can represent the possible solutions in a nice binary notation

00=000000
01=000001=2^0
02=000010=2^1
03=000011=$2^1 + 2^0$
04=000100=2^2
…
07=000111=$2^2 + 2^1 + 2^0$
08=001000=2^3
…
15=001111=$2^3 + 2^2 + 2^1 + 2^0$
16=010000=2^4
…
31=011111=$2^4 + 2^3 + 2^2 + 2^1 + 2^0$
32=100000=2^5



We trick the monster by adding the constraint the x must be integer – **but we refresh binary numbers first**

BARCELONA SCHOOL OF MANAGEMENT

Never met binary?

$00=000000$
$01=000001=2^0$
$02=000010=2^1$
$03=000011=2^1+2^0$
$04=000100=2^2$
…
$07=000111=2^2+2^1+2^0$
$08=001000=2^3$
…
$15=001111=2^3+2^2+2^1+2^0$
$16=010000=2^4$
…
$31=011111=2^4+2^3+2^2+2^1+2^0$
$32=100000=2^5$
$64 \ =1000000=2^6$
$128=10000000=2^7$

Take a way to write a number

CCXXVI

This is conceptually **far** from

226

units

tens

hundreds

While this is conceptually close to

11100010

$32=2^5$

$64=2^6$

$128=2^7$

BARCELONA SCHOOL OF MANAGEMENT

Never met binary?

0.1=1/2
0.01=1/4
0.001=1/8

0.111=0.875

11.001=3.125

...

Take a way to write a number

CCXXVI

This is conceptually **far** from

226

units

tens

hundreds

While this is conceptually close to

11100010

$32=2^5$

$64=2^6$

$128=2^7$

BARCELONA SCHOOL OF MANAGEMENT

Then we have to cook some rules, e.g.

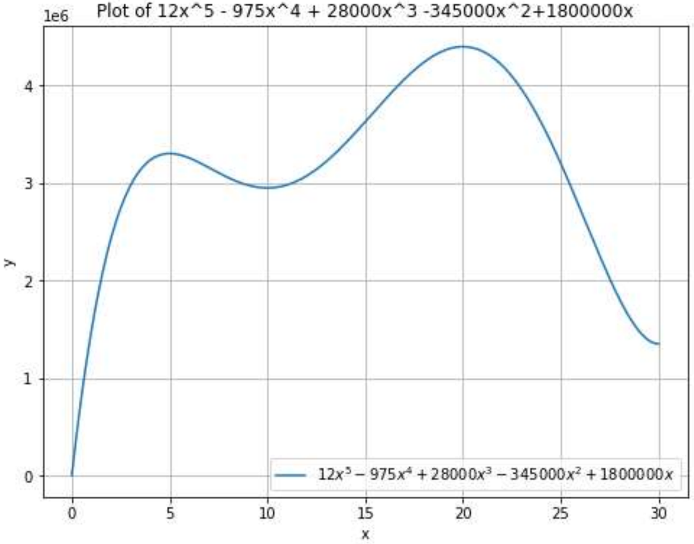
Starting population: 10 individuals (for this problem)

Choose the five fittest and the two lest fit for **matching** (coupling randomly)

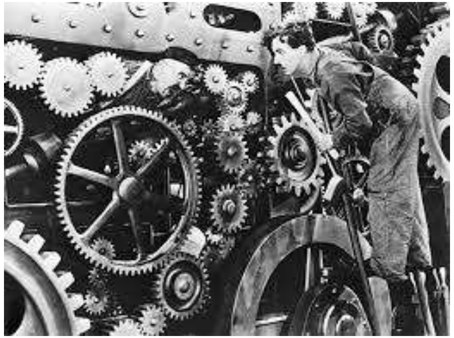
Switching the genes: keep repeated genes and switch the different ones

Mutation rate 1/10

Stopping rule after five iterations without improvements



Lots of moving parts; these are the choices suggested in Hillier but other choices are possible



Source: Charlie Chaplin's Modern Times

upf. BARCELONA SCHOOL OF MANAGEMENT

One possible way of matching

Parents
011000
110101

Repeated genes are passed to the next generation

Children
x1xx0x
x1xx0x

How to choose the missing x's? **Russian roulette**

x1xx0x
x1xx0x

Generate a random number between 0 and 1; if between 0 and .4999 replace the first x with a zero, if between .5 and 1 replace it with a one

Where the name comes from – from revolver to spinning wheel9

Source: From movie The Deer Hunter, Source: Wikipedia commons

Source: https://www.gettyimages.es

Source: https://wordwall.net/

BARCELONA SCHOOL OF MANAGEMENT

How to choose the x's? **Russian roulette**

x1xx0x
x1xx0x

Generate a random number between 0 and 1; if between 0 and .4999 replace the first x with a zero, if between .5 and 1 replace it with a one

This was done and we have children

011000
010101

How to mutate 10% of this? Russian roulette: if the random number is between 0 and 0.0999 choose to mutate, otherwise no mutations and pass to the next couple;

Russian roulette also to decide which gene to change; if there are 12 (6+ 6) candidate genes (two parent of five genes each), **how do we do this time?**

If between 0 and $\frac{1}{12}$ mutate the first gene of the first child, in between $\frac{1}{12}$ and $\frac{2}{12}$ the second gene of the fist child ⋯ if between $\frac{11}{12}$ and 1 the last gene of the second child

Source: https://www.gettyimages.es

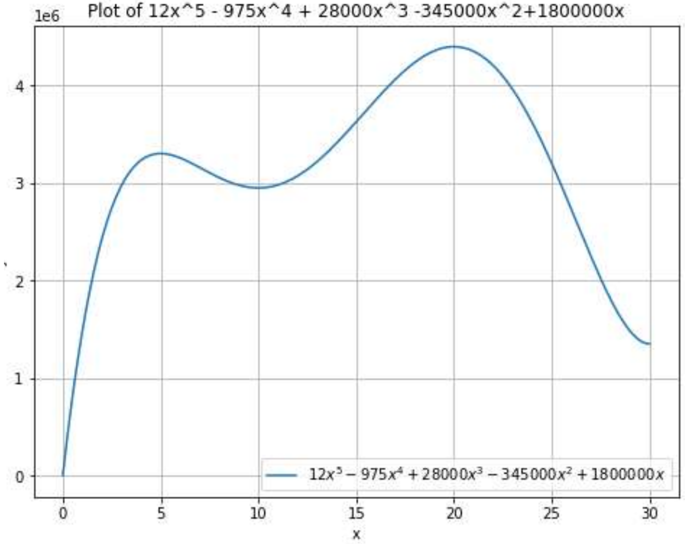
Source: https://wordwall.net/

BARCELONA SCHOOL OF MANAGEMENT

TABLE 14.7 Application of the genetic algorithm to the integer nonlinear programming example through (*a*) the initialization step and (*b*) iteration 1

	Member	Initial Population	Value of *x*	Fitness
	1	0 1 1 1 1	15	3,628,125
	2	0 0 1 0 0	4	3,234,688
	3	0 1 0 0 0	8	3,055,616
	4	1 0 1 1 1	23	3,962,091
(a)	5	0 1 0 1 0	10	2,950,000
	6	0 1 0 0 1	9	2,978,613
	7	0 0 1 0 1	5	3,303,125
	8	1 0 0 1 0	18	4,239,216
	9	1 1 1 1 0	30	1,350,000
	10	1 0 1 0 1	21	4,353,187

	Member	Parents	Children	Value of *x*	Fitness
	10	1 0 1 0 1	0 0 1 0 1	5	3,303,125
	2	0 0 1 0 0	1 0 0 0 1	17	4,064,259
(b)	8	1 0 0 1 0	1 0 0 1 1	19	4,357,164
	4	1 0 1 1 1	1 0 1 0 0	20	4,400,000
	1	0 1 1 1 1	0 1 0 1 1	11	2,980,637
	6	0 1 0 0 1	0 1 1 1 1	15	3,628,125



All this is very nice to code; here the starting and first iteration from the Hillier book

And the convergence if rapid for this simple case, good results already at the fist iteration

upf. BARCELONA SCHOOL OF MANAGEMENT

The procedure can of course be applied to non-integer numbers, as these can as well be written in binary notation

Exercise: write 412 in binary

412=110011100

256y 128y 64n 32n 16y 8y 4y 2n 1n

I start by 256 because the next power of 2 (512) is too big; **get 1**

412-256=156; so I can fit in 128; **get 11**

The difference is 28, so I cannot fit a 64: **get 110**

Cannot fit a 32, **get 1100**

Can fit 16,8,4 add to 28 **get 1100111**

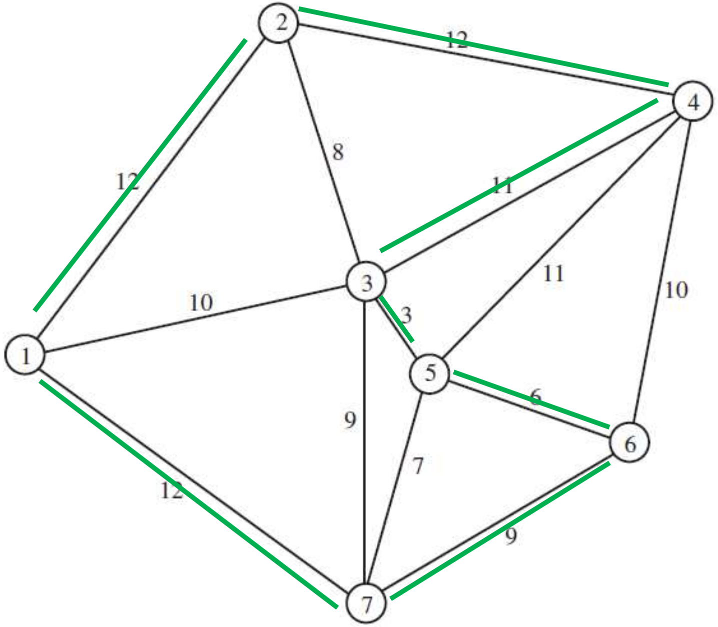
No need of 2 and 1 the last two powers, **get 110011100**

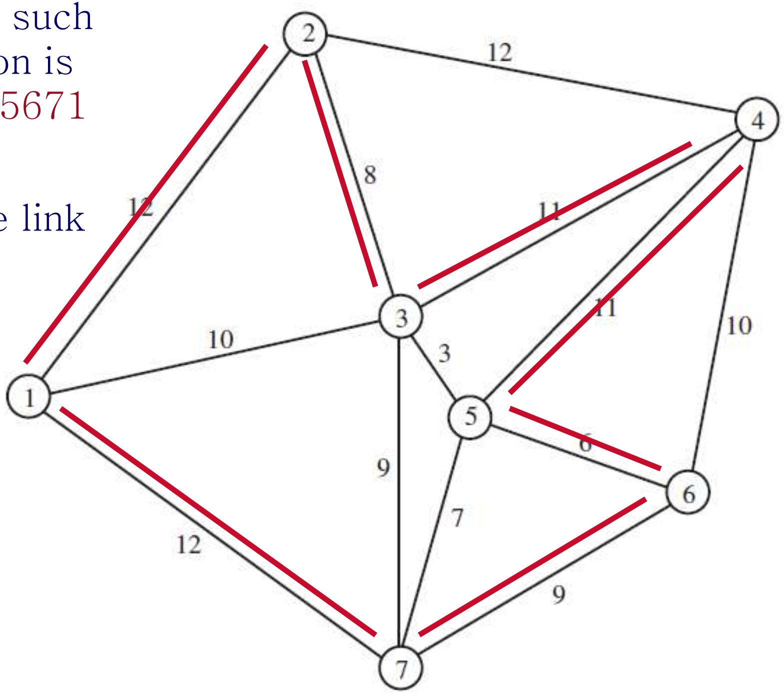
Source: https://simpsons.fandom.com/wiki/Bart_Gets_Famous

BARCELONA SCHOOL OF MANAGEMENT

Genetic algorithms can also be applied to problems such as the traveling salesman; in this case the population is constituted by candidate trajectories, such as 12345671 and 12435671

A child of this couple can inherit the link 2–3 or the link 2–4 …





Homework

1) Read pages 208-225 of the Mann book (saved in Campus as file Fact-Binom.pdf) and solve exercises 5.41, 5.48. It is not forbidden to use Excel.

2) Solve Hillier online book problem 12.1.3 page 534, only question (a) Formulate a BIP model for this problem.

3) Solve Hillier online book problem 12.1-4. page 534, only question (a) Formulate a BIP model for this problem.

4) Solve Hillier online book problem 12.3-1. page 535, only question (a) Formulate a BIP model for this problem..

Thank you

www.andreasaltelli.eu
https://orcid.org/0000-0003-4222-6975
@AndreaSaltelli@mstdn.social
https://www.youtube.com/channel/UCz26ZK04xchekUy4GevA3DA

upf. BARCELONA SCHOOL OF MANAGEMENT