Máster Universitario en Administración y Dirección de Empresas

Full Time MBA

Quantitative methods for decision making

Professor Andrea Saltelli

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Elements of quantification for decision making with emphasis on operation research

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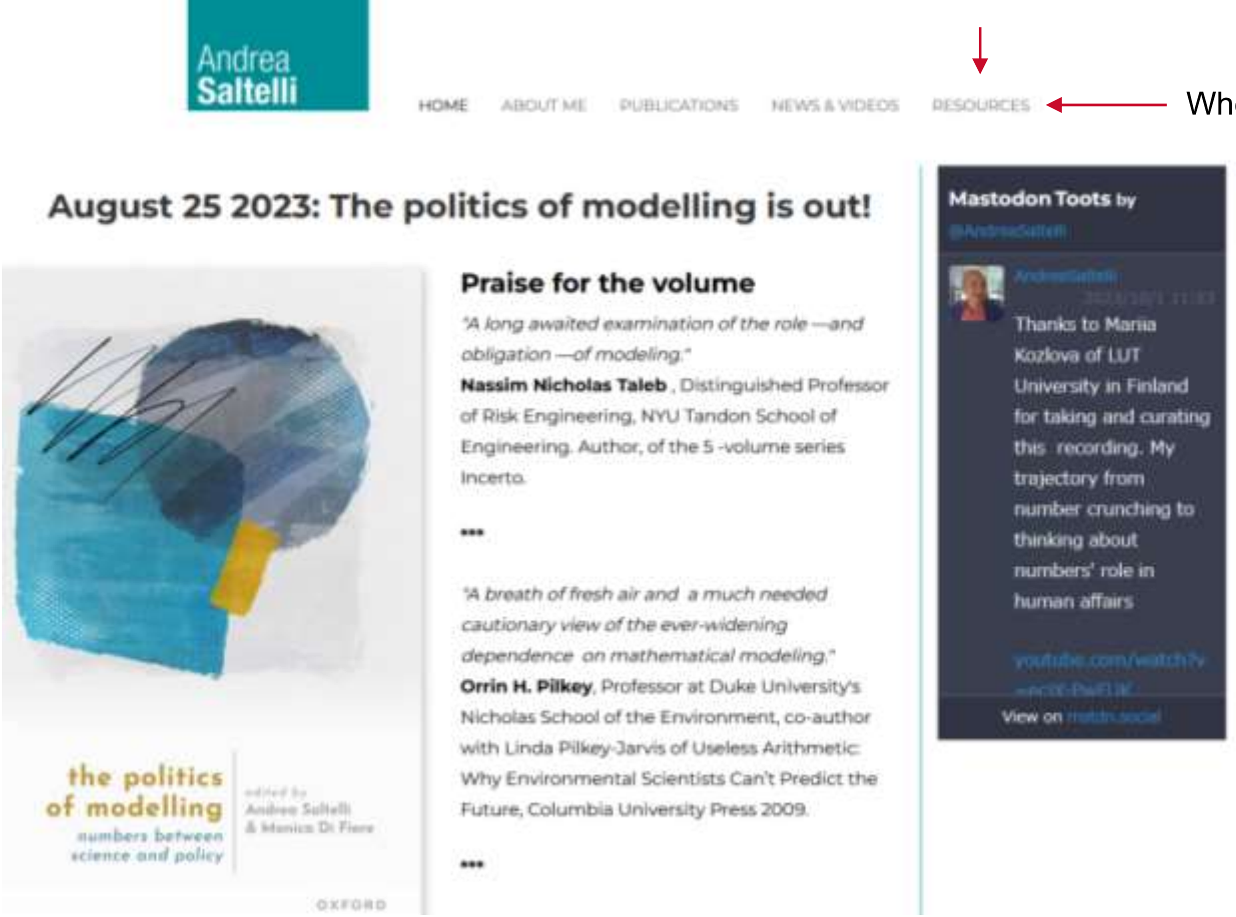

Where to find this talk

The talk is also at

https://ecampus.bsm.upf.edu/,

where you find additional reading material

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In this set of slides:

12 The Transportation Problem

13 The Assignment Problems (sketched)

14 Network Optimization Models

15 Integer Programming

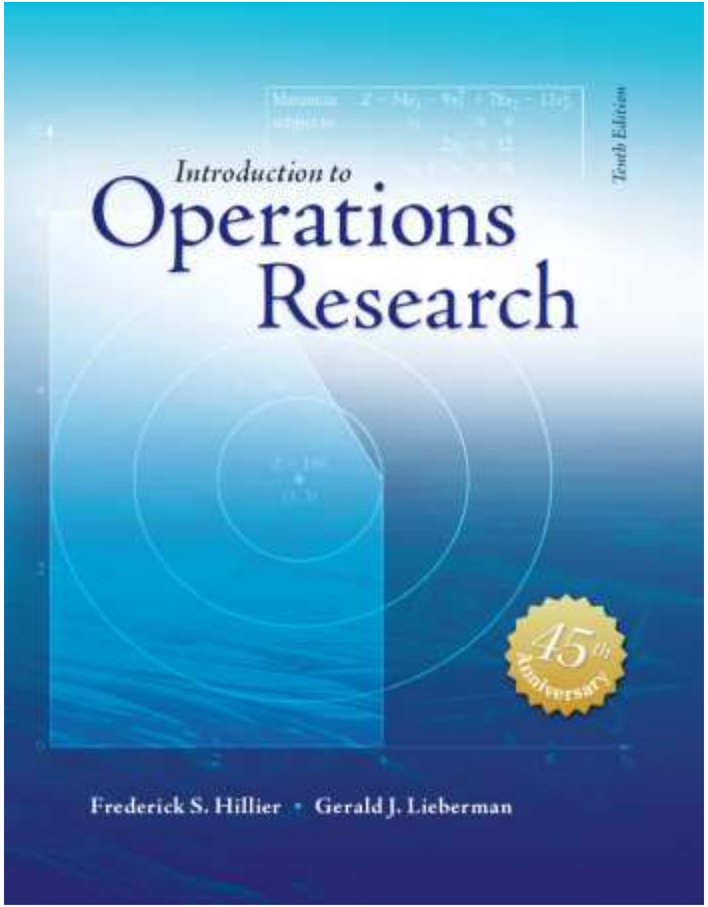
12.

The Transportation problem

Framing of the problem, assumptions and properties of the solution. Hillier 2014, chapter 9.

Where to find this book:

https://www.dropbox.com/sh/ddd48a8jguinbcf/AABF0s4eh1lPLVxdx0pes-Ofa?dl=0&preview=Introduction+ to+ Operations+ Research+ -
+ Frederick+ S.+ Hillier.pdf



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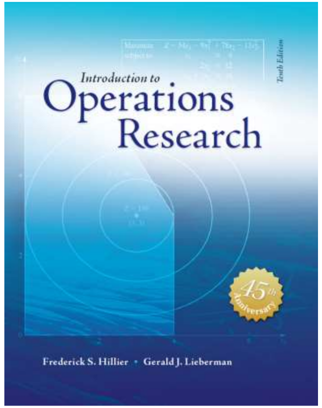
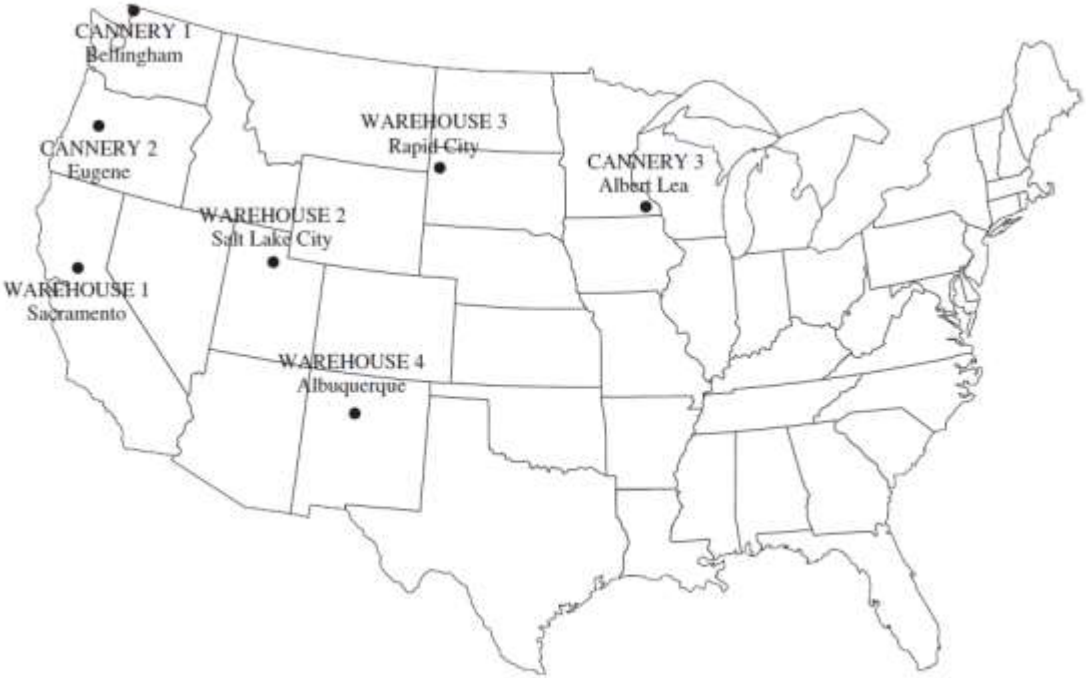
A prototype example of a Transportation Problem: shipping canned peas from canneries to warehouses



Three canneries and four warehouses



■ **FIGURE 9.1**
Location of canneries and warehouses for the P & T Co. problem.

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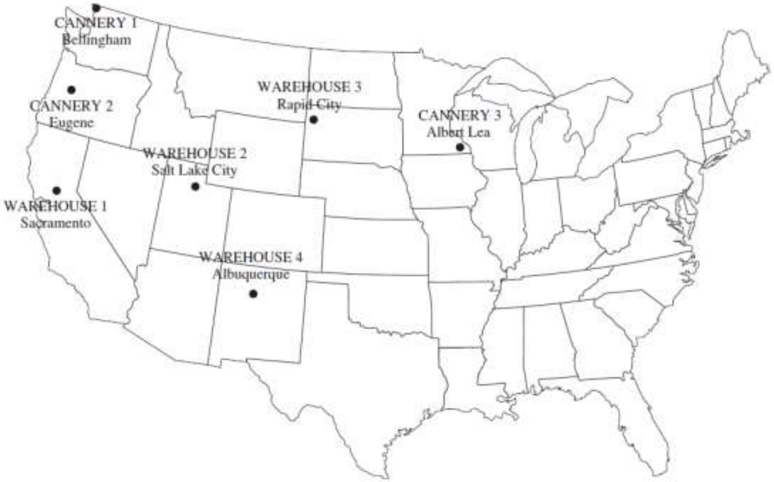
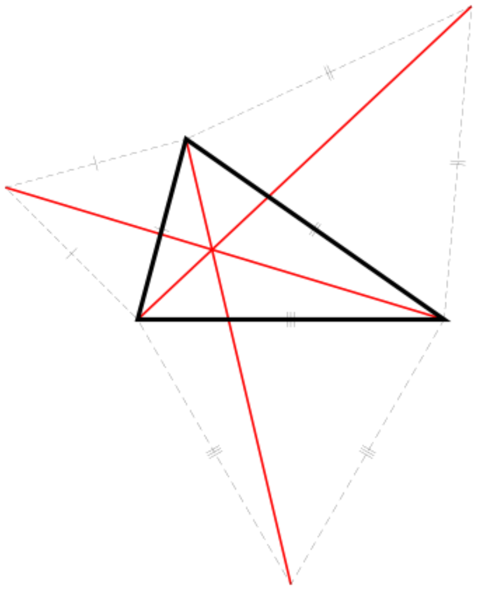
A prototype example: shipping truckloads of canned peas from canneries to warehouses

Source: Wikipedia Commons



■ **FIGURE 9.1**
Location of canneries and warehouses for the P & T Co. problem.

An old type of problem, recall the Torricelli and Fermat point



Source: Wikipedia Commons

1.Construct an equilateral triangle on each of the sides
2. From each of the farmost vertex draw a line the opposite vertex of the original triangle.
3. Where the three lines intersect is the Torricelli-Fermat point.

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A prototype example: shipping canned peas from canneries to warehouses; this table contains all the information; where are the geographical distances?

TABLE 9.2 Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	**2**	**3**	**4**	**Output**
	1	464	513	654	867	75
Cannery	2	352	416	690	791	125
	3	995	682	388	685	100
Allocation		80	65	70	85	

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In modern linear programming the geography can be made to disappear

Here it is replaced by costs per truckload per season

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A prototype example: shipping canned peas from canneries to warehouses

We know how much moving truckloads costs

… and how much each warehouse should be provided with

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Subject to cannery constraints

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Minimize or maximize? → Minimize

What? → Total shipping cost; decision variable $x_{i,j}, i = 1,2,3; j = 1,2,3,4$ member of truckloads from cannery i to warehouse j

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TABLE 9.2 Shipping data for P & T Co.

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		Warehouse				
		1	2	3	4	Output
	1	464	513	654	867	75
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$$\text{Minimize total shipping cost } Z = 464\, x_{1,1} + 513\, x_{1,2} + 654\, x_{1,3} + 867\, x_{1,4} + 352\, x_{2,1} + 416\, x_{2,2} + 690\, x_{2,3} + 791\, x_{2,4} + 995 x_{3,1} + 682\, x_{3,2} + 388\, x_{3,3} + 685\, x_{3,4}$$

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		Shipping Cost ($) per Truckload				
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	3	995	682	388	685	100
Allocation		80	65	70	85	

Subject to cannery constraints

$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 75$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 125$$
$$x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = 100$$

and warehouse constrains

$$x_{1,1} + x_{2,1} + x_{3,1} = 80$$
$$x_{1,2} + x_{2,2} + x_{3,2} = 65$$
$$x_{1,3} + x_{2,3} + x_{3,3} = 70$$
$$x_{1,4} + x_{2,4} + x_{3,4} = 85$$

$$x_{i,j} \geq 0 \ (i = 1,2,3; j = 1,2,3,4)$$

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■ **TABLE 9.2** Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
		1	2	3	4	Output
Cannery	1	464	513	654	867	75
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Allocation		80	65	70	85	

Anything noticeable about these two sets of numbers?

Supply and demand balance out at 300

$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 75$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 125$$
$$x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = 100$$

These constraints can be written as a distinct pattern that is characteristic of the Transportation and Assignment Problem

TABLE 9.3 Constraint coefficients for P & T Co.

	Coefficient of:												
	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	
A =	1	1	1	1									Cannery constraints
					1	1	1	1					
									1	1	1	1	
	1				1				1				Warehouse constraints
		1				1				1			
			1				1				1		
				1				1				1	

$$x_{1,1} + x_{2,1} + x_{3,1} = 80$$
$$x_{1,2} + x_{2,2} + x_{3,2} = 65$$
$$x_{1,3} + x_{2,3} + x_{3,3} = 70$$
$$x_{1,4} + x_{2,4} + x_{3,4} = 85$$

■ **TABLE 9.2** Shipping data for P & T Co.

		Shipping Cost ($) per Truckload				
		Warehouse				
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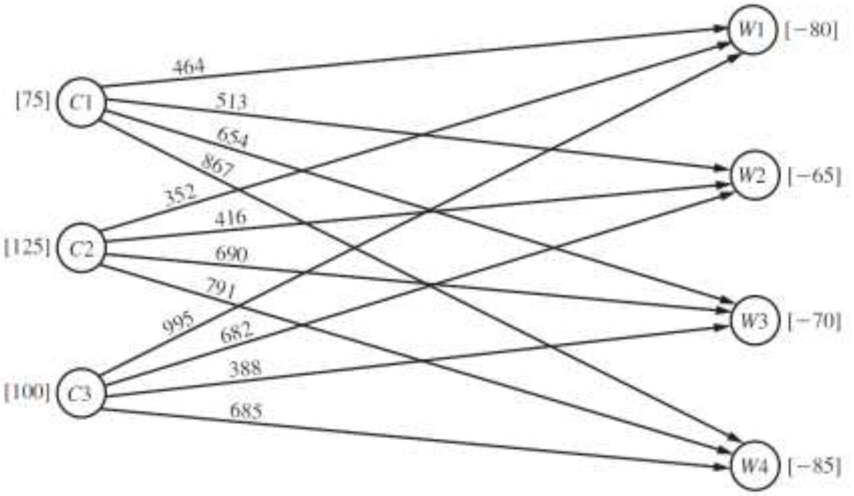
Or as a graph/network representation



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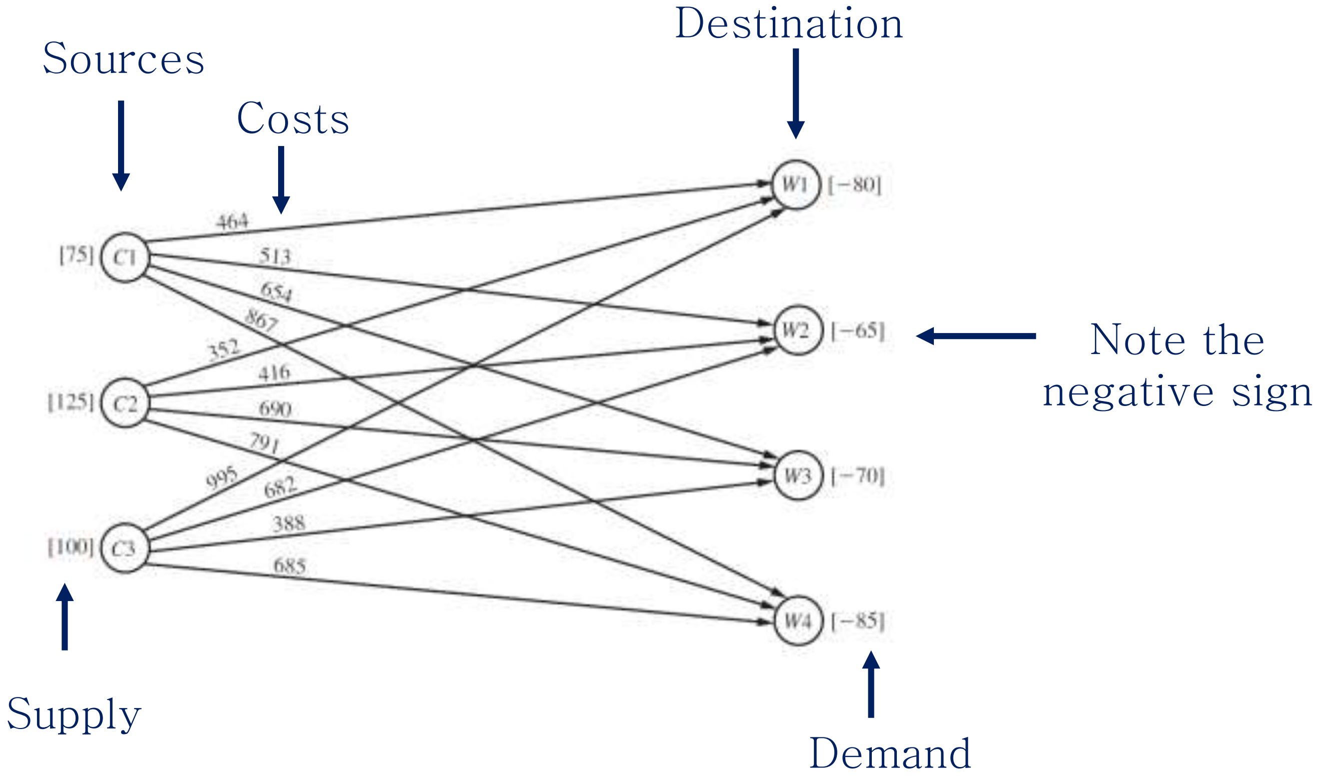
Terminology of the Transportation and Assignment Problem



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$$x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 75$$
$$x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 125$$
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$$x_{1,3} + x_{2,3} + x_{3,3} = 70$$
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The = sign (instead of ≤≥) in the supply and demand represents the **requirement assumption** of the Transportation and Assignment Problem: supply and demand are fixed

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Allocation		80	65	70	85	

Minimize total shipping cost $Z =$

$$= 464\, x_{1,1} + 513\, x_{1,2} + 654\, x_{1,3} + 867\, x_{1,4}$$
$$+ 352\, x_{2,1} + 416\, x_{2,2} + 690\, x_{2,3} + 791\, x_{2,4}$$
$$+ 995 x_{3,1} + \; 682\, x_{3,2} + 388\, x_{3,3} + 685\, x_{3,4}$$

The **cost assumption**: distributing units from any source to any destination is proportional to the number of units distributed; if c_{ij} is the unit cost and x_{ij} the number of units, the cost is simply $c_{ij}x_{ij}$

The requirements assumption is typic of transportation problem, while the **cost assumption** we should know already

What are the assumptions we studied already?

Assumptions of linear programming

Proportionality: The contribution of each activity to the value of the objective function Z is proportional to the level of the activity x_j increase in the objective function Z, as represented by the $c_j x_j$ terms

$$\text{Maximize} \quad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

Additivity: Every function in a linear programming model (whether the objective function or the function on the left-hand side of a functional constraint) is the sum of the individual contributions of the respective activities

Divisibility: Decision variables in a linear programming model are allowed to have any values, including noninteger values, that satisfy the functional and nonnegativity constraints. Thus, these variables are not restricted to just integer values. Since each decision variable represents the level of some activity, it is being assumed that the activities can be run at fractional levels

When a decision variable **must** be an integer, it becomes a case of integer programming

Certainty: The value assigned to the parameters (the a_j^i's, b_i's, and c_j's) of a linear programming model are assumed to be known constants

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Whether or not actual transportation is involved, any problem in the format of this table that obeys the requirement and cost assumption is a transportation problem

TABLE 9.5 Parameter table for the transportation problem

		Cost per Unit Distributed				
		Destination				
		1	2	⋯	n	Supply
Source	1	c_{11}	c_{12}	⋯	c_{1n}	s_1
	2	c_{21}	c_{22}	⋯	c_{2n}	s_2
	⋮	⋯	⋯	⋯	⋯	⋮
	m	c_{m1}	c_{m2}	⋯	c_{mn}	s_m
Demand		d_1	d_2	⋯	d_n	

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Compact formulation for a problem with m sources s and n destinations d:

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to source and demand constraints

$$\sum_{j=1}^{n} x_{ij} = s_i \quad \text{for} \quad i = 1,2,\dots m$$

$$\sum_{i=1}^{m} x_{ij} = d_j \text{ for } j = 1,2,\dots n$$

$$x_{ij} \geq 0 \text{ for } (i = 1,2,\dots m;\ j = 1,2,\dots n)$$

TABLE 9.5 Parameter table for the transportation problem

		Cost per Unit Distributed				
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		1	**2**	···	***n***	**Supply**
Source	1	c_{11}	c_{12}	···	c_{1n}	s_1
	2	c_{21}	c_{22}	···	c_{2n}	s_2
	⋮					⋮
	m	c_{m1}	c_{m2}	···	c_{mn}	s_m
Demand		d_1	d_2	···	d_n	

The property to be kept in mind here is that a transportation problem will have feasible solution if and only if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

(supply and demand balance out as in the example)

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Compact formulation for a problem with m sources s and n destinations d:

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to

$\sum_{i=1}^{m} x_{ij} = d_j$ for $j = 1,2, \ldots n$

$\sum_{j=1}^{n} x_{ij} = s_i$ for $i = 1,2, \ldots m$

$x_{ij} \geq 0$ for $(i = 1,2, \ldots m;\ j = 1,2, \ldots n)$

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

(supply and demand balance out)

The integer solutions property: For transportation problems where every s_i and d_i have an integer value, all basic feasible (BF) solutions (including an optimal one) also have integer values

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■ **TABLE 9.2** Shipping data for P & T Co.

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Cannery	1	464	513	654	867	75
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Allocation		80	65	70	85	

Optimal solution with Excel Solver

0	20	0	55
80	45	0	0
0	0	70	30

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Omer would like 2 pints of home brew today and an additional 7 pints of home brew tomorrow. Dick is willing to sell a maximum of 5 pints total at a price of $3.00 per pint today and $2.70 per pint tomorrow. Harry is willing to sell a maximum of 4 pints total at a price of $2.90 per pint today and $2.80 per pint tomorrow. Omer wishes to know what his purchases should be to minimize his cost while satisfying his thirst requirements.

Formulate this problem as a *transportation problem* by constructing the appropriate parameter table

	Today	Tomorrow	
Dick	3.	2.70	5
Harry	2.90	2.80	4
Tom/day	2	7	

What would you do being Omer?

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13.

The Assignment problem

A brief sketch. Hillier 2014, chapter 9.

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The assignment problem is a special type of linear programming problem where **assignees** are being assigned to perform **tasks**

Charles Chaplin's Modern Times, source http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

1. The number of assignees and the number of tasks are the same.
2. Each assignee is to be assigned to exactly one task.
3. Each task is to be performed by exactly one assignee.
4. There is a cost c_{ij} associated with assignee i, $(i = 1,2,\ldots n)$ performing task j, $(j = 1,2,\ldots n)$.
5. The objective is to determine how all n assignments should be made to minimize the total cost ⋯ but

Source: Wikipedia Commons

?

Charles Chaplin's Modern Times, source
http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

In fact, the assignment problem is just a special type of transportation problem where the **sources now are assignees** and the **destinations now are tasks** and where:

Number of sources m = number of destinations n,
Every supply $s_i = 1$,
Every demand $d_j = 1$

Number of sources m = number of destinations n,
Every supply $s_i = 1$,
Every demand $d_i = 1$

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1,2, \ldots n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for} \quad i = 1,2, \ldots n$$

$$x_{ij} \geq 0 \text{ for } (i = 1,2, \ldots n;\ j = 1,2, \ldots n)$$

Plus

x_{ij} = binary $(0 \text{ or } 1)$ for $(i = 1,2, \ldots n;\ j = 1,2, \ldots n)$

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$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1,2,\ldots n$$ ← Each task must be served

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for} \quad i = 1,2,\ldots n$$ ← Each assignee must have work

$$x_{ij} \geq 0 \text{ for } (i = 1,2,\ldots n;\ j = 1,2,\ldots n)$$

Plus

x_{ij} = binary (0 *or* 1) for

$(i = 1,2,\ldots n;\ j = 1,2,\ldots n)$

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Thus assignment and transportation share the same useful properties in terms of existence of integer solutions

Source: Wikipedia Commons

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Charles Chaplin's Modern Times, source
http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

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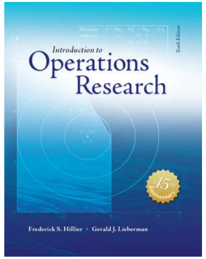
Assignment and transportation have same network representation



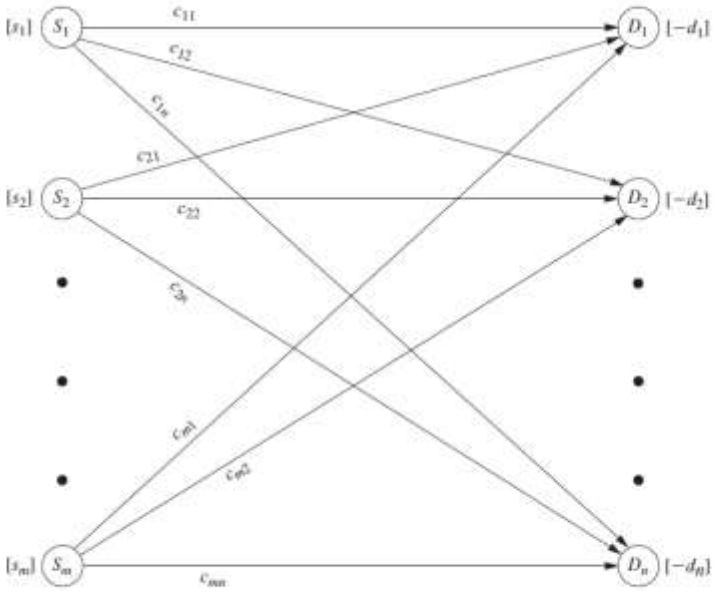


FIGURE 9.3
Network representation of the transportation problem.

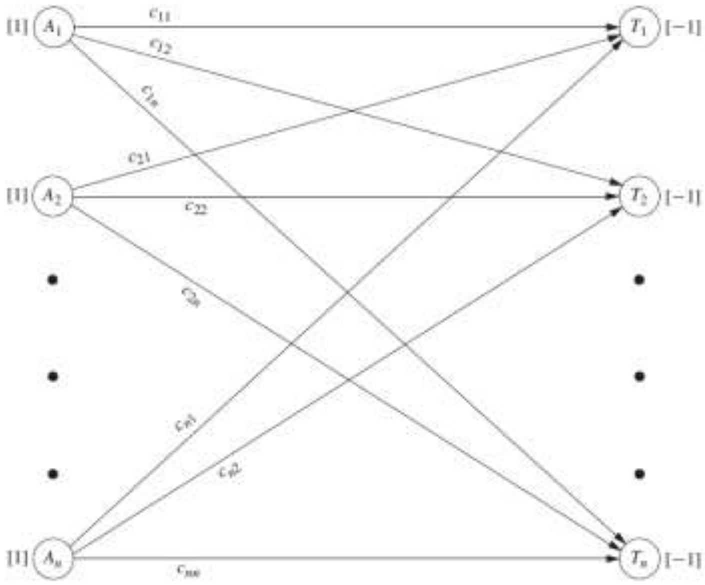


FIGURE 9.5
Network representation of the assignment problem.

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A typical problem offered in the book locating three machine among four facilities, with different cost per machine / facility

TABLE 9.24 Materials-handling cost data ($) for Job Shop Co.

		Location			
		1	**2**	**3**	**4**
Machine	1	13	16	12	11
	2	15	—	13	20
	3	5	7	10	6

		Task (Location)			
		1	**2**	**3**	**4**
Assignee (Machine)	1	13	16	12	11
	2	15	*M*	13	20
	3	5	7	10	6

Machine 2 cannot go to location 2, so a very large cost M in entered in the empty cell

A typical problem offered in the book locating three machine among four facilities, with different cost per machine / facility

TABLE 9.24 Materials-handling cost data ($) for Job Shop Co.

		Location			
		1	2	3	4
Machine	1	13	16	12	11
	2	15	—	13	20
	3	5	7	10	6

TABLE 9.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	2	3	4
Assignee (Machine)	1	13	16	12	11
	2	15	*M*	13	20
	3	5	7	10	6
→	4(*D*)	0	0	0	0

Since assignees and tasks must be equal a dummy machine is introduced

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A typical problem offered in the book locating three machine among four facilities, with different cost per machine / facility

TABLE 9.25 Cost table for the Job Shop Co. assignment problem

		Task (Location)			
		1	**2**	**3**	**4**
	1	13	16	12	11
Assignee	2	15	*M*	13	20
(Machine)	3	5	7	10	6
	4(*D*)	0	0	0	0

Can you guess the solution “by inspection?”

Machine 1 to location 4
Machine 2 to location 3
Machine 3 to location 1
The algorithms (not described here) would assign the dummy machine 4 to location 2

14.

Network Optimization Models

More network problems: shortest-path problem, the minimum spanning tree problem, maximum flow problem. Hiller 2014, chapter 10.

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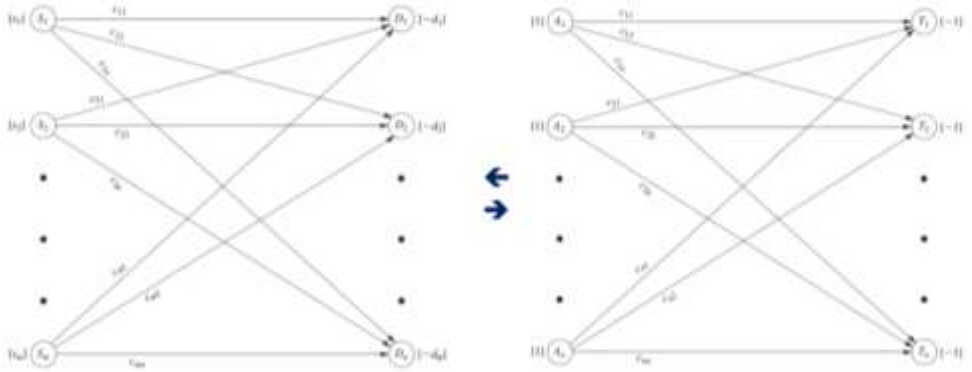
Many network optimization models are special types of linear programming problems – e.g. the transportation problem and the assignment problem

See their network representations

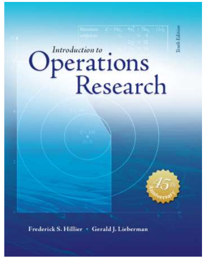
Assignment and transportation have same network representation

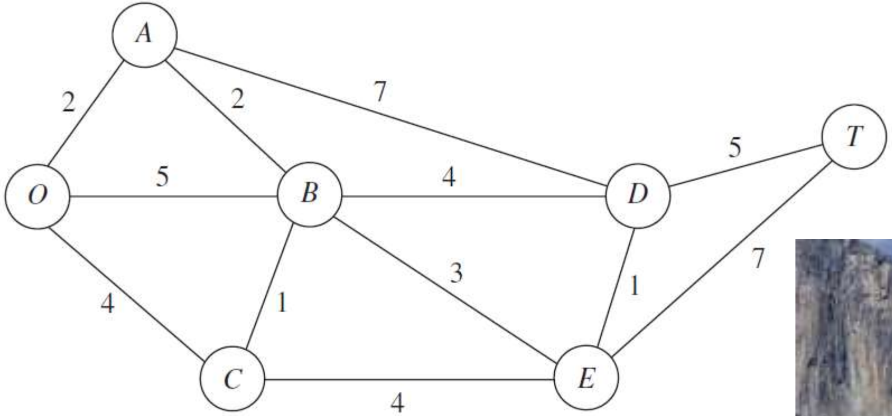


■ **FIGURE 9.3**
Network representation of the transportation problem.

■ **FIGURE 9.5**
Network representation of the assignment problem.

Our new prototype problem – the "Seervada Park" road system





■ **FIGURE 10.1**
The road system for Seervada Park.

Source: https://www.klook.com/en-US/activity/28218-yosemite-park-giant-sequoia-day-tour-san-francisco/?

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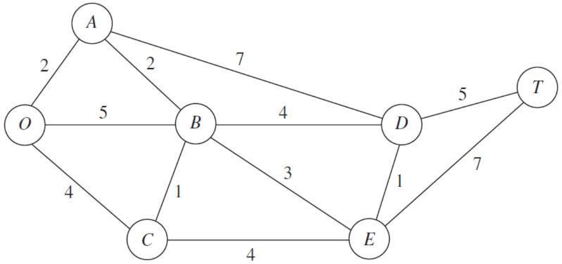
Three practical problems

- **Shortest path** from entrance O to scenic point T
- Minimum length of telephone lines covering all tracks (**minimum spanning tree**)
- **Maximum flow** of mini-trains carrying non trekkers from entrance O to scenic point T



Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

Source: https://www.klook.com/en-US/activity/28218-yosemite-park-giant-sequoia-day-tour-san-francisco/?

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Some terminology: nodes (or vertices), arcs (or links or edges or branches)

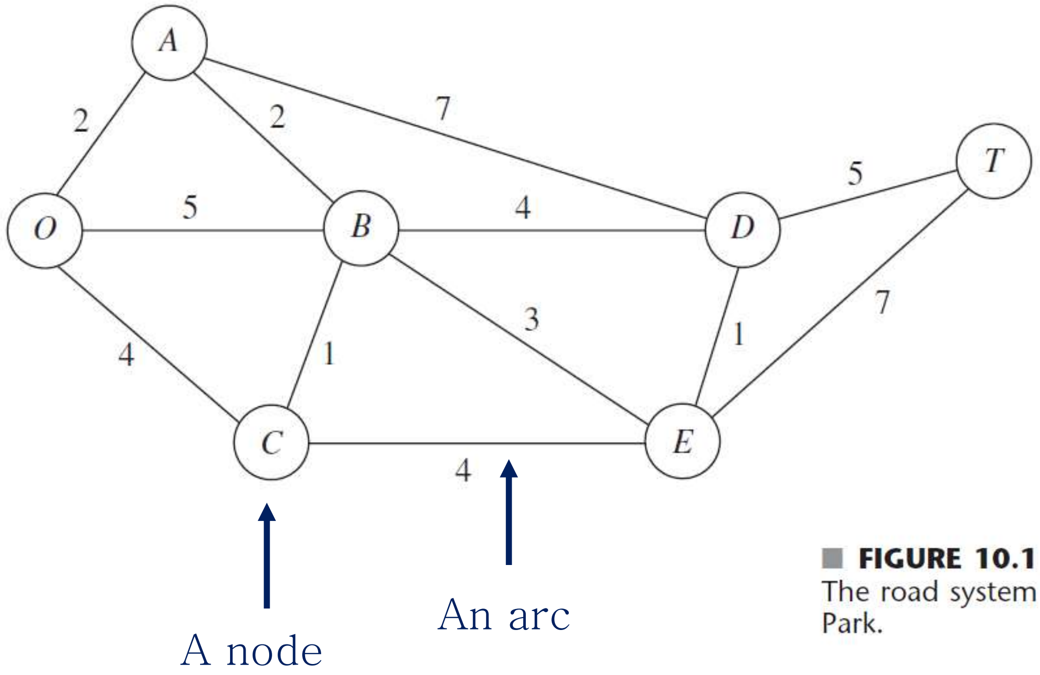
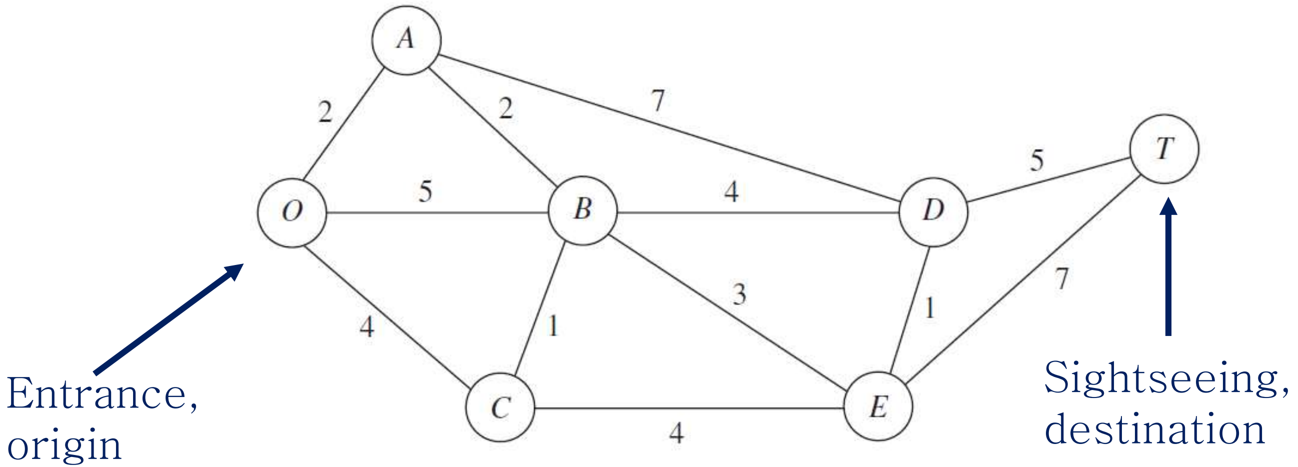


FIGURE 10.1
The road system for Seervada Park.


A
O
B
D
T
C
E
2
2
7
5
4
5
4
1
3
1
7
4
Entrance, origin
Sightseeing, destination

The trains trough the park represent a type of ‘flow’ through the arcs

Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

TABLE 10.1 Components of typical networks

Nodes	Arcs	Flow
Intersections	Roads	Vehicles
Airports	Air lanes	Aircraft
Switching points	Wires, channels	Messages
Pumping stations	Pipes	Fluids
Work centers	Materials-handling routes	Jobs

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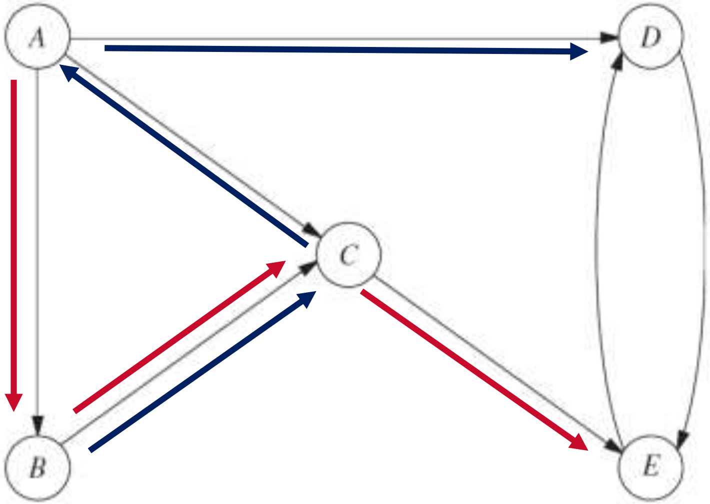
More terminology:

Directed arcs (flow only in one directions) and undirected arcs or link, (flow in both directions)
Networks can also be directed (only directed arcs) or undirected
A **path** trough nodes can be directed when every step from node *i* to node *j* is in the direction of *j*.

A→B →C→E = directed path

B→C→A→D = undirected path



Note that our park has no arrows, in is hence made of undirected arcs

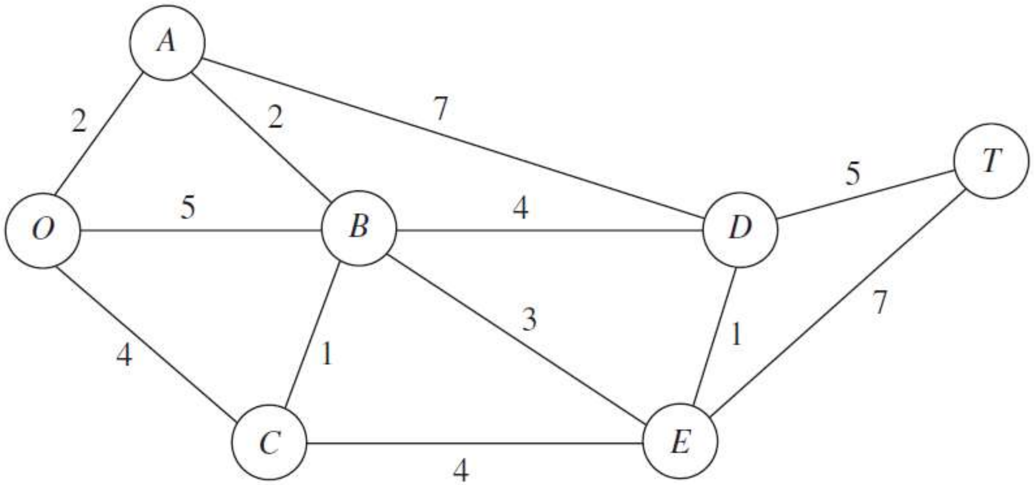
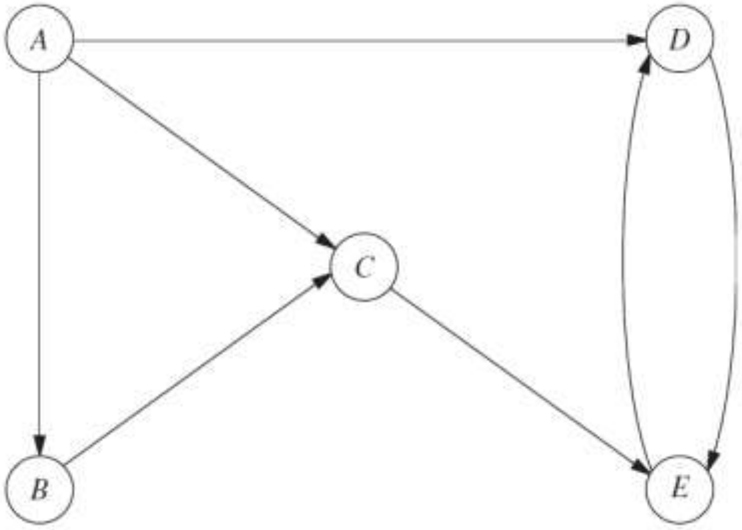


FIGURE 10.1
The road system for Seervada Park.

More terminology: a **cycle** is a path starting and ending in the same node



A **directed cycle** contains only directed arcs

D→E→D is a directed cycle

A→B→C→A is not a directed cycle

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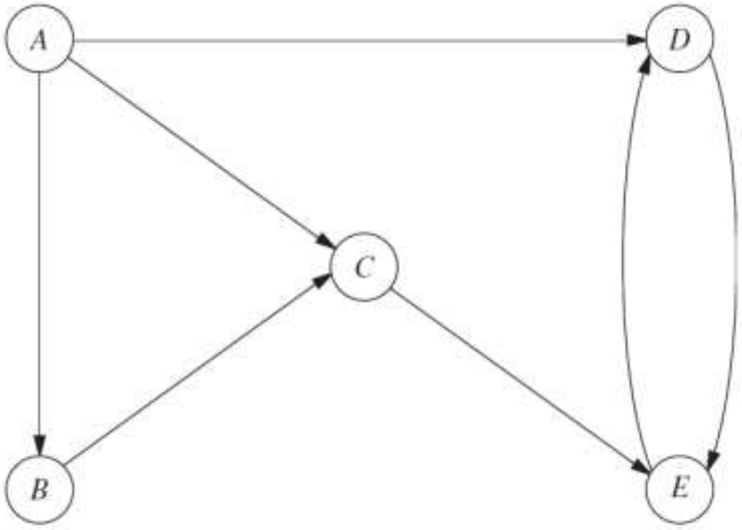
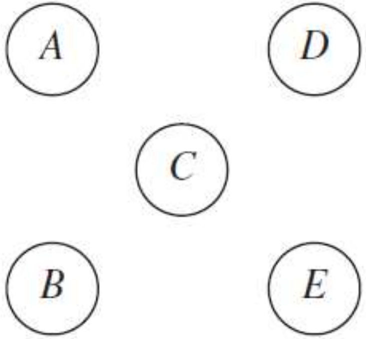
More terminology: starting from bare nodes, **trees** can be grown



A network;
stripping the
arc one gets ⋯



⋯ bare nodes

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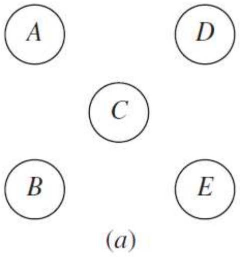
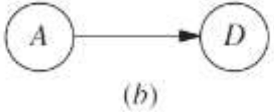
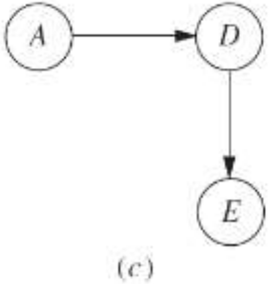
Starting from bare nodes, **trees** can be grown



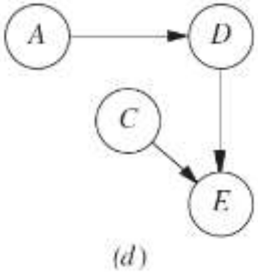
(a) bare nodes



(b) Tree with one arc

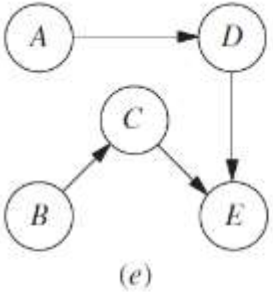


(c) Tree with two arcs

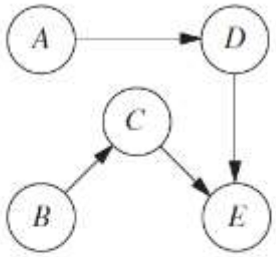


(d) Tree with three arcs

(e) Spanning tree: all nodes connected by directed arcs



(e) Spanning tree: all nodes connected by directed arcs

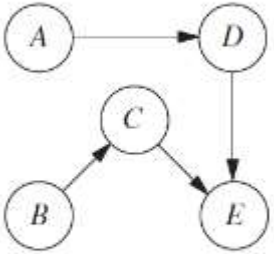


(e)

A spanning tree connects *n* nodes with *n-1* directed arcs

A spanning tree is a **connected network** without unconnected nodes

(e) Spanning tree: all nodes connected by directed arcs



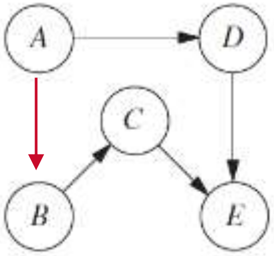
(e)

A spanning tree connects *n* nodes with *n-1* directed arcs

A spanning tree is a **connected network** without unconnected arcs

n-1 is both the **minimum** number of arcs needed and the **maximum** one, as adding one arc would generate an undirected **cycle**

Adding e.g. arc A→C closes the loop but generates undirected cycles



(e)

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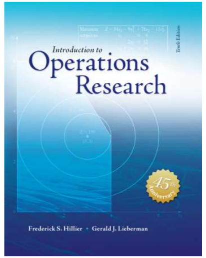
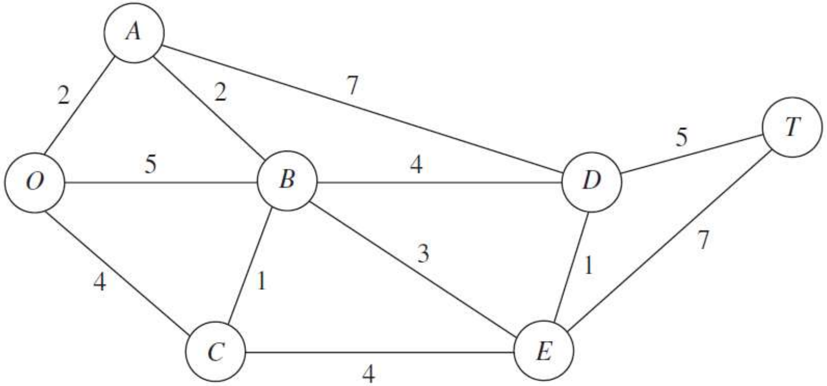
We are now ready to tackle the **shortest path problem**

Ramon Casas and Pere Romeu on a Tandem, Barcelona. Source: Wikipedia Commons

“Consider an undirected and connected network with two special nodes called the origin and the destination. Associated with each of the links (undirected arcs) is a nonnegative distance. The objective is to find the shortest path (the path with the minimum total distance) from the origin to the destination”





Let’s learn by doing, on our test case: the mission is to go from the entrance O to the scenic point T

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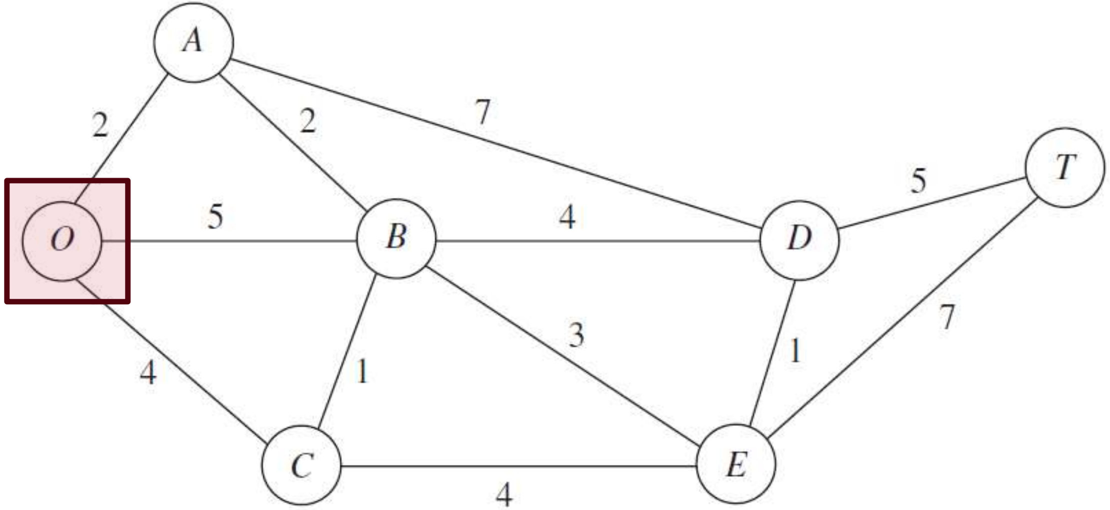
Algorithm for the Shortest-Path Problem



Theory: Objective of nth iteration: Find the nth nearest node to the origin (to be repeated for $n = 1, 2, \ldots$ until the nth nearest node is the destination.

Practice: the nearest note to O is A

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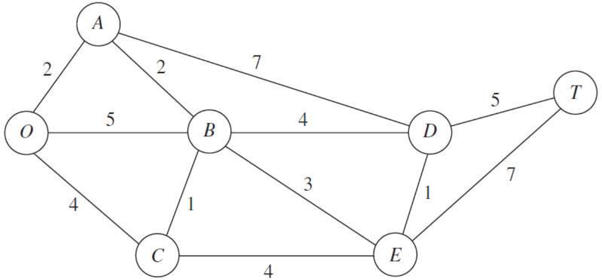

Theory: Objective of nth iteration: Find the nth nearest node to the origin (to be repeated for $n = 1, 2, ...$ until the nth nearest node is the destination.
Practice: the nearest note to O is A

TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*

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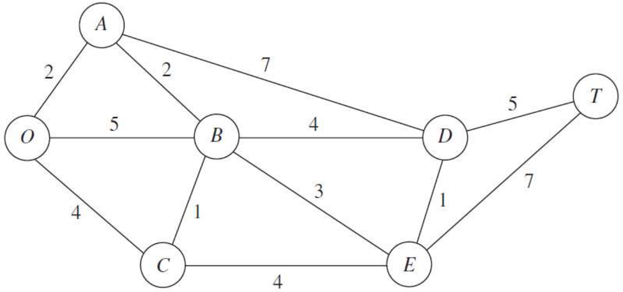
Algorithm for the Shortest-Path Problem



Theory: Input needed for nth iteration: $n-1$ nearest nodes to the origin (solved for at the previous iterations), including their shortest path and distance from the origin. (These nodes, plus the origin, will be called solved nodes; the others are unsolved nodes)

Theory: Candidates for nth nearest node: Each solved node that is directly connected by a link to one or more unsolved nodes provides one candidate — the unsolved node with the shortest connecting link to its solved node is taken

Theory: Candidates for nth nearest node: Each solved node (O, A now) that is directly connected by a link to one or more (nearest) unsolved nodes (C, B respectively) provides one candidate — the unsolved node with the shortest connecting link to this solved node. (Ties provide additional candidates)

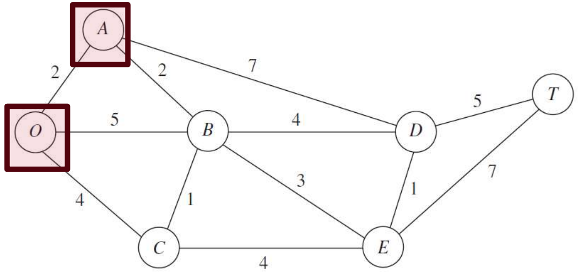


TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*

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Theory: Calculation of nth nearest node: For each such solved node and its candidate, add the distance between them and the distance of the shortest path from the origin to this solved node. The candidate with the smallest such total distance is the nth nearest node (ties provide additional solved nodes – as in this case C and B with 4 miles), and its shortest path is the one generating this distance

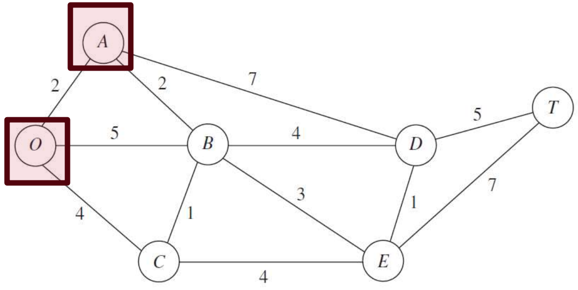

TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

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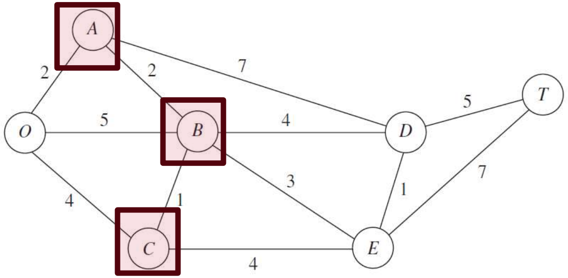
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*

The solved nodes are now A, B, C, and the closest nodes are D, E
(E is closest for both B and C)
E wins as 4th closest node (7 miles)



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2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*

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The solved nodes closest to an unsolved note are now A, B, E, and for all the closest node is D
D wins as 5th closest node (8 miles)

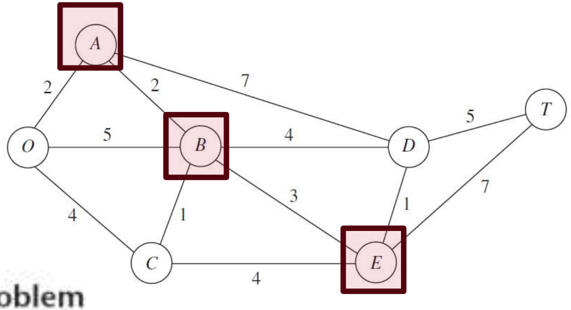

TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*
5	*A* *B* *E*	*D* *D* *D*	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	*D* *D*	8 8	*BD* *ED*

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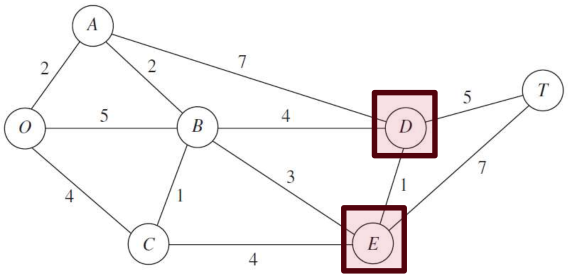
The solved nodes closest to an unsolved note are now D, E, and for both the closest node is the target destination T ; T wins as 6th closest node (13 miles)



■ **TABLE 10.2** Applying the shortest-path algorithm to the Seervada Park problem

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2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*
5	*A* *B* *E*	*D* *D* *D*	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	*D* *D*	8 8	*BD* *ED*
6	*D* *E*	*T* *T*	8 + 5 = 13 7 + 7 = 14	*T*	13	*DT*

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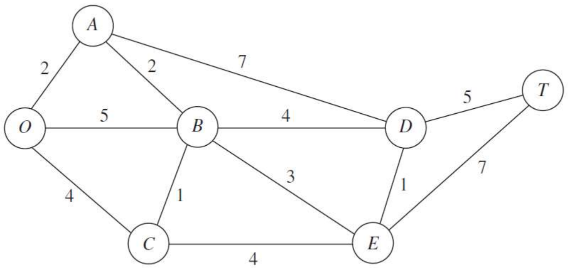
TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*
5	*A* *B* *E*	*D* *D* *D*	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	*D* *D*	8 8	*BD* *ED*
6	*D* *E*	*T* *T*	8 + 5 = 13 7 + 7 = 14	*T*	13	*DT*



… and the minimum distance is recorded

Note how at each step the distance for the various candidate is computed…

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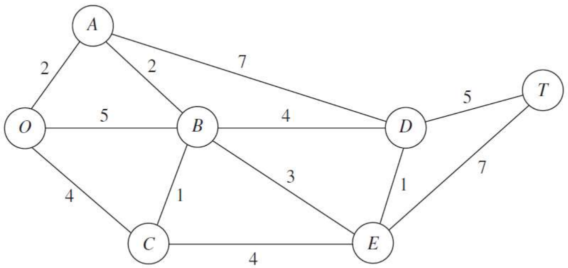
TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	nth Nearest Node	Minimum Distance	Last Connection
1	O	A	2	A	2	OA
2, 3	O A	C B	4 2 + 2 = 4	C B	4 4	OC AB
4	A B C	D E E	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	E	7	BE
5	A B E	D D D	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	D D	8 8	BD ED
6	D E	T T	8 + 5 = 13 7 + 7 = 14	T	13	DT

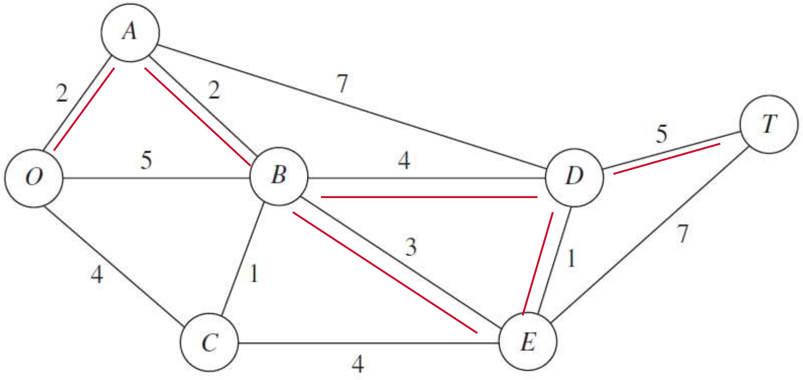


We now move backword, from the destination to the origin
T→D→B→A→O
or
T→D→E→B→A→O
Both with 13 miles

Hence the solution:
O→A→B→D→T or
O→A→B→E→D→T

TABLE 10.2 Applying the shortest-path algorithm to the Seervada Park problem

n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	*n*th Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	2	*A*	2	*OA*
2, 3	*O* *A*	*C* *B*	4 2 + 2 = 4	*C* *B*	4 4	*OC* *AB*
4	*A* *B* *C*	*D* *E* *E*	2 + 7 = 9 4 + 3 = 7 4 + 4 = 8	*E*	7	*BE*
5	*A* *B* *E*	*D* *D* *D*	2 + 7 = 9 4 + 4 = 8 7 + 1 = 8	*D* *D*	8 8	*BD* *ED*
6	*D* *E*	*T* *T*	8 + 5 = 13 7 + 7 = 14	*T*	13	*DT*



T→D → D→B→A→O

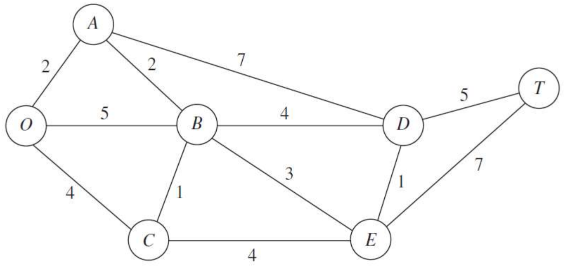
T→D → D→E→B→A→O

Perhaps clearer in this tree formulation?

Hence the solution:
O→A→B→E→D→T
or
O→A→B→D→T

Three practical problems

- **Shortest path** from entrance O to scenic point T ← Solved
- Minimum length of telephone lines covering all tracks (**minimum spanning tree**)
- **Maximum flow** of mini-trains carrying non trekkers from entrance O to scenic point T



Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

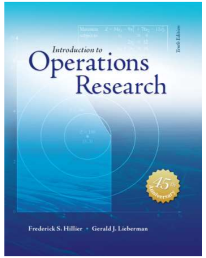
Source: https://www.klook.com/en-US/activity/28218-yosemite-park-giant-sequoia-day-tour-san-francisco/?

The Minimum Spanning Tree problem

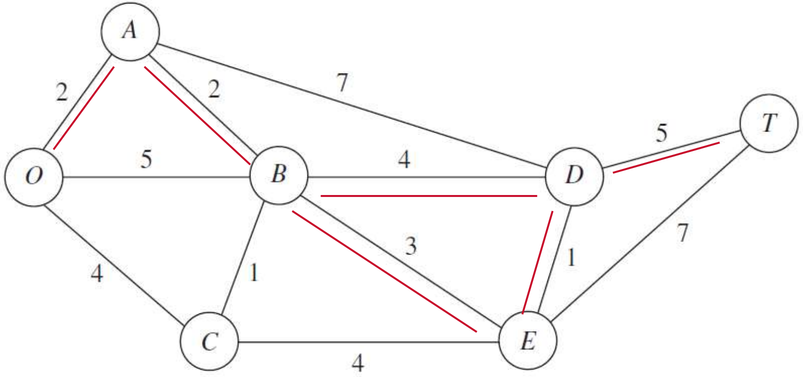
Source: https://eu.palmbeachdailynews.com/story/entertainment/house-home/2019/12/15/palm-beach-gardening-help-save-planet-by-planting-these-native-trees/2079095007/

The Minimum Spanning Tree problem

For the shortest-path problem, we were looking for links that provide a path between the origin and the destination. We now just look for a minimum set of links that connect all nodes



Could this be a spanning tree?



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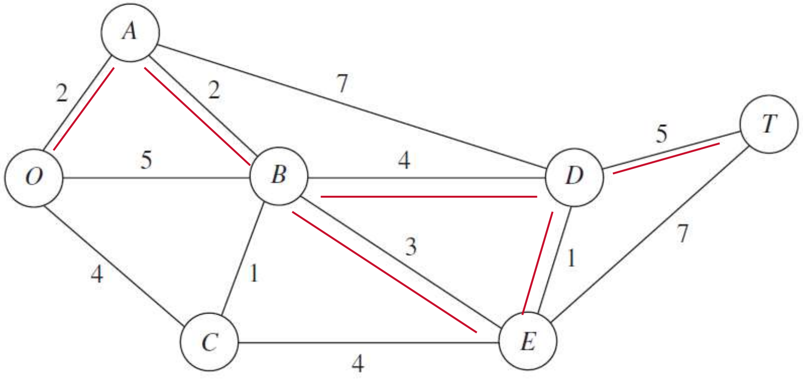
No as a spanning tree provides a path between each pair of nodes. ***n*** nodes will take ***n-1*** links

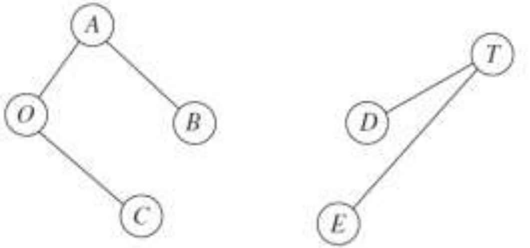
➔ Design the network by inserting enough links to satisfy the requirement that there be a path between every pair of nodes; The objective is to satisfy this requirement in a way that minimizes the total length of the links

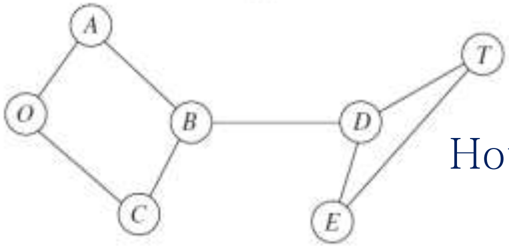
Could this in red be a spanning tree?



➔ Design the network by inserting enough links to satisfy the requirement that there be a path between every pair of nodes; The objective is to satisfy this requirement in a way that minimizes the total length of the links

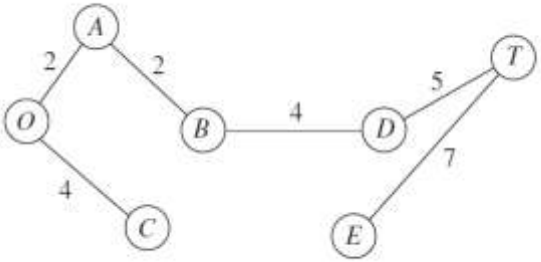
Could this be a spanning tree?





How about this?

Perhaps this?



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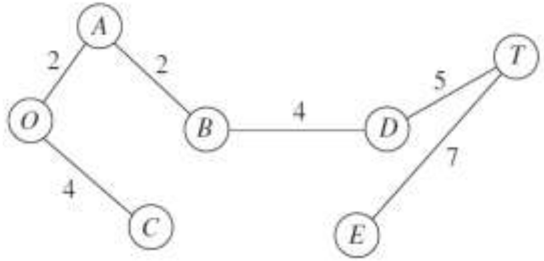
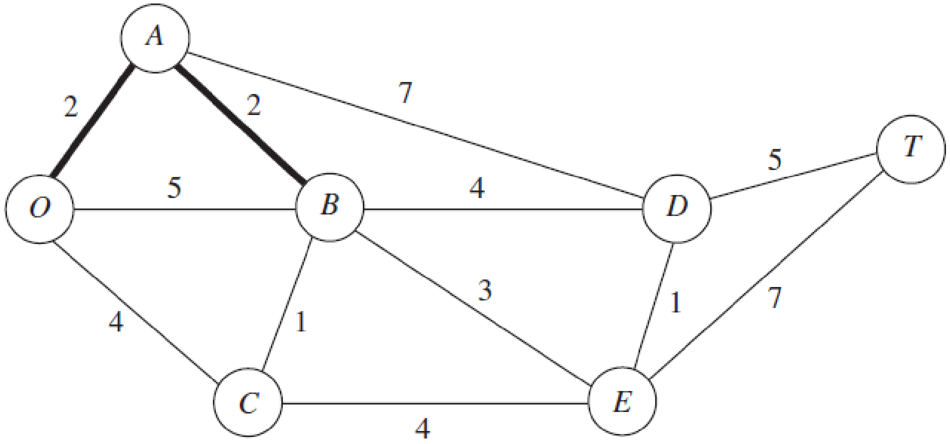


All **7** nodes connected with **6** link

The strategy

- Select arbitrarily a node
- Identify closest unconnected node
- Branch on ties (try both)



Select arbitrarily a node e.g. A
Identify closest unconnected node O or B
Branch on ties (try both)

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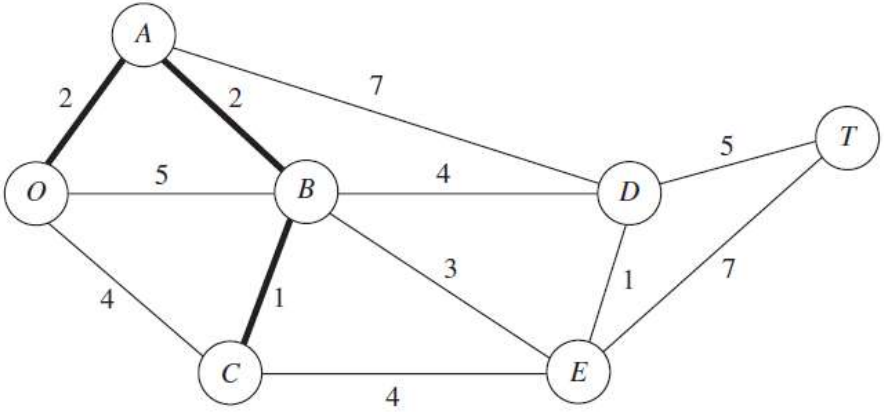
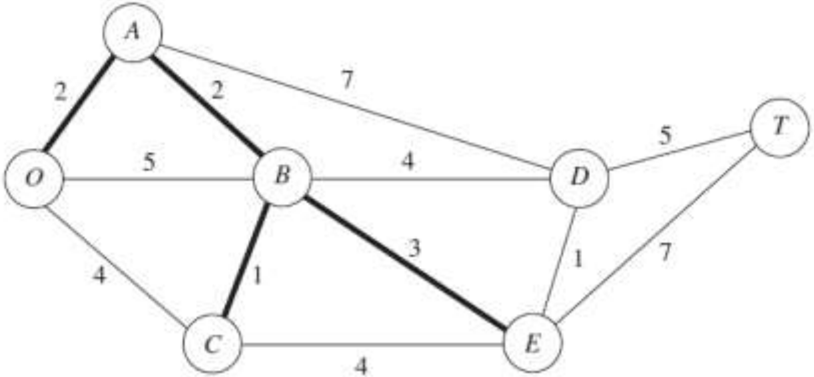
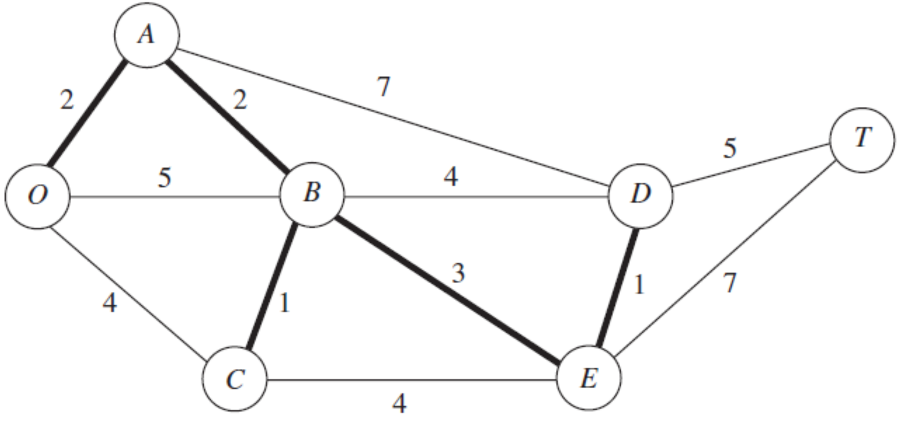


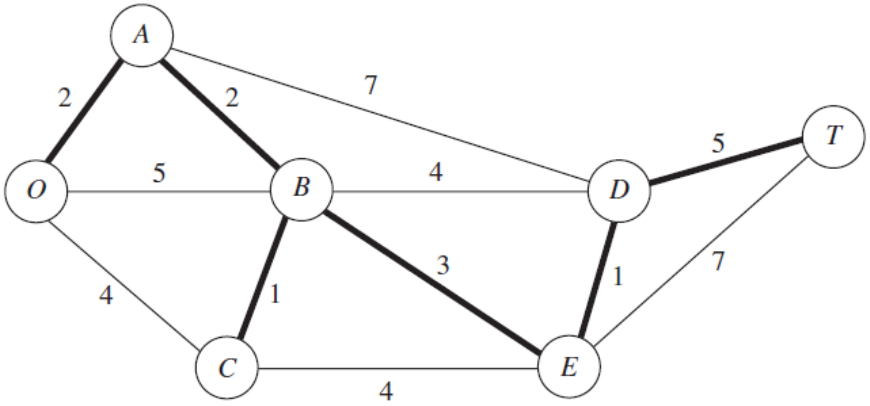
Identify closest unconnected node C



Identify closest unconnected node E



Identify closest unconnected node D



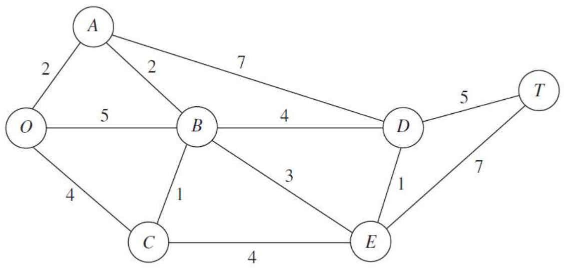
Here our spanning tree

Three practical problems

- **Shortest path** from entrance O to scenic point T
- Minimum length of telephone lines covering all tracks (**minimum spanning tree**)

← Solved

- **Maximum flow** of mini-trains carrying non trekkers from entrance O to scenic point T

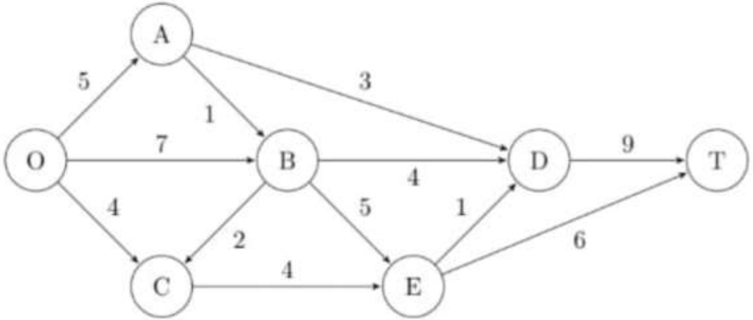


Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

Source: https://www.klook.com/en-US/activity/28218-yosemite-park-giant-sequoia-day-tour-san-francisco/?

We are now left with the last problem to solve: **Maximum flow** of mini-trains carrying non trekkers from entrance O to scenic point T

Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/



Maximum flow problem

"Typical kinds of applications of the maximum flow problem:

1. Maximize the flow through a company's distribution network from its factories to its customers.
2. Maximize the flow through a company's supply network from its vendors to its factories.
3. Maximize the flow of oil through a system of pipelines.
4. Maximize the flow of water through a system of aqueducts.
5. Maximize the flow of vehicles through a transportation network." (Hillier pp.387-388)

Source: https://www.livescience.com/61862-why-phantom-traffic-jams-happen.html

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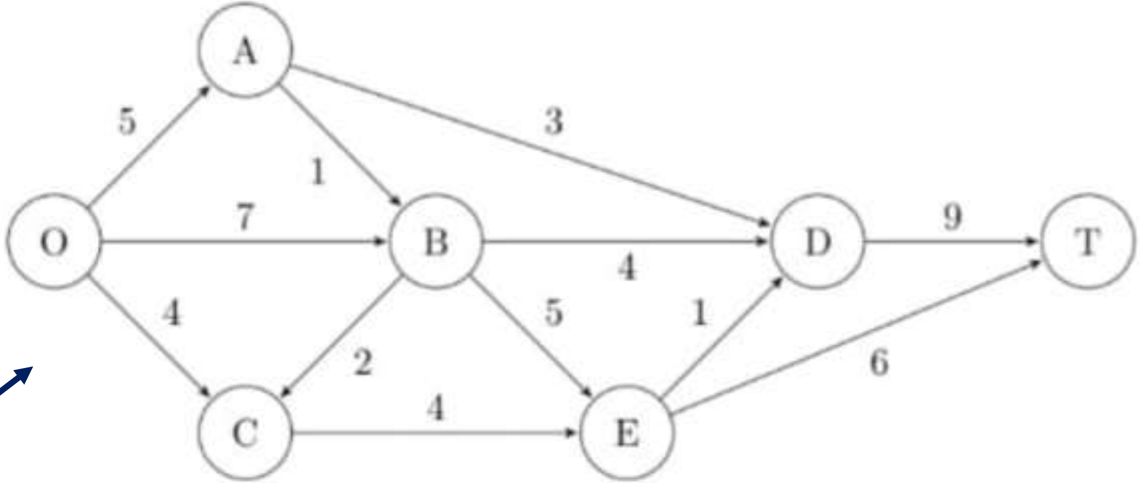
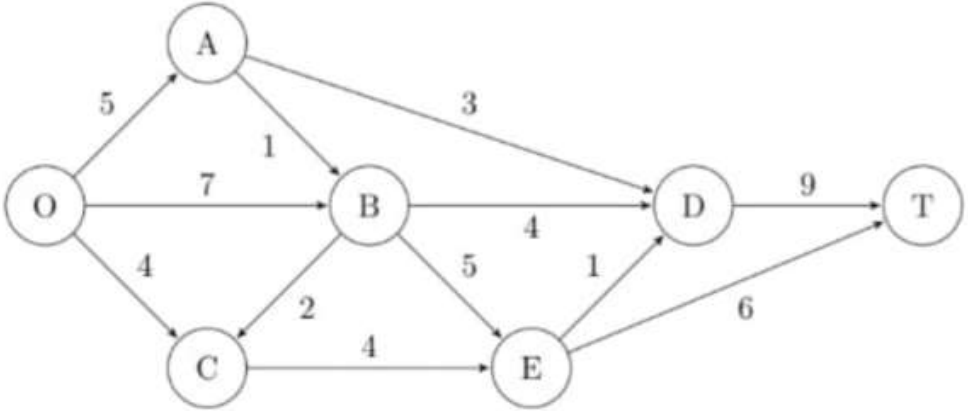
Maximum flow problem

Also here we proceed by a stepwise algorithm by 'pumping' items along preselected paths and recording changes. Numbers now represent maximum capacities

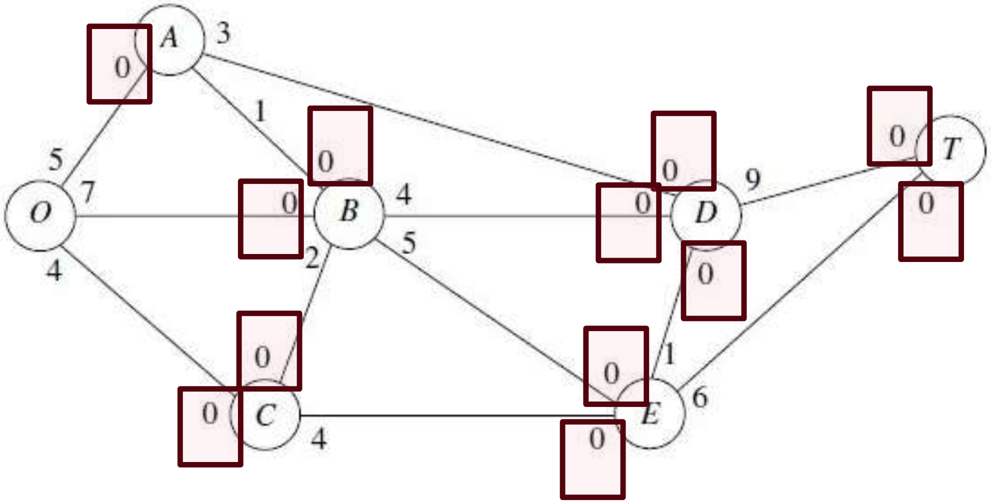
Warning: figures 10.6 and 10.7 in the online version are wrong, the others are right https://www.dropbox.com/sh/ddd48a8jguinbcf/AABF0s4eh1lPLVxdx0pes-Ofa?dl=0&preview=Introduction+to+Operations+Research+-+Frederick+S.+Hillier.pdf

This is right





Nothing has moved yet, and we note this by putting zeros **before** the node



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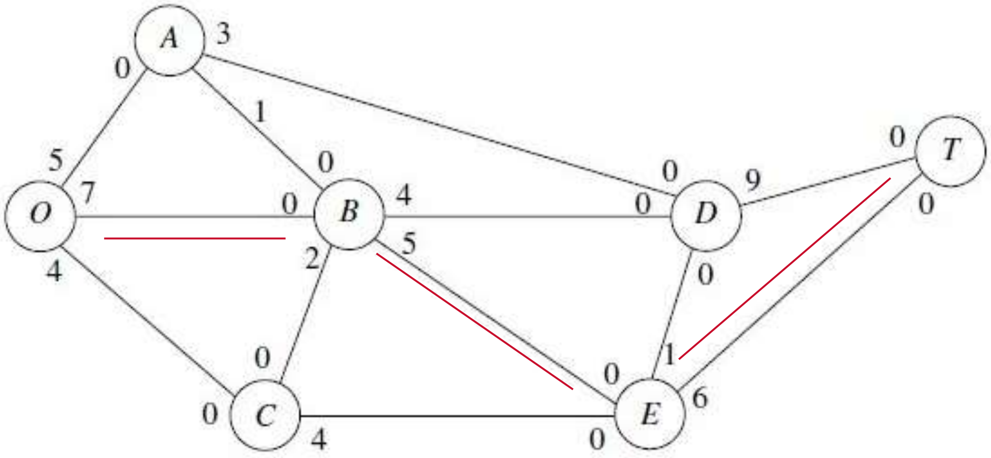


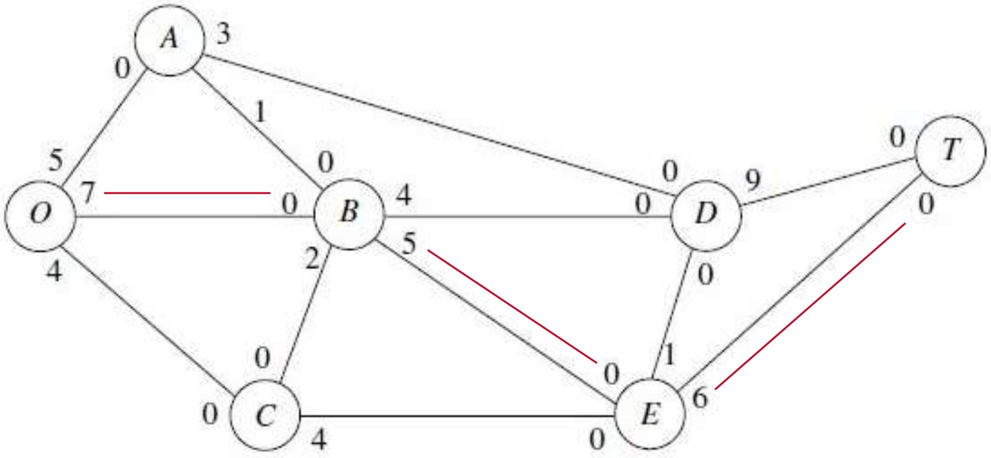
An augmenting path is a directed path from the source to the sink in the residual network such that every arc on this path has strictly positive residual capacity; for example

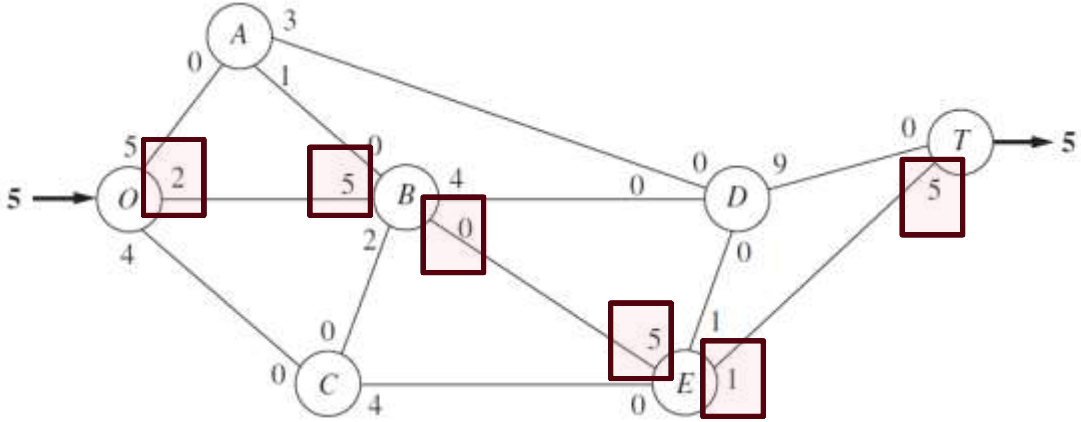
$$O \to B \to E \to T$$

is an augmenting path, still at full capacity.

Chose now the smallest **residual capacity** on this path – among **7,5,6 →** **5** is the smallest. Move five through this path, noting what happens

The capacity of link *BE* is now exhausted



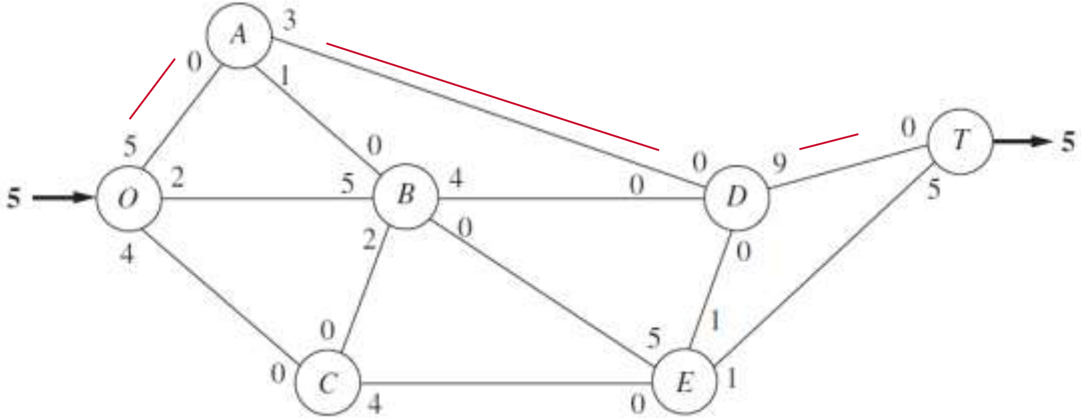


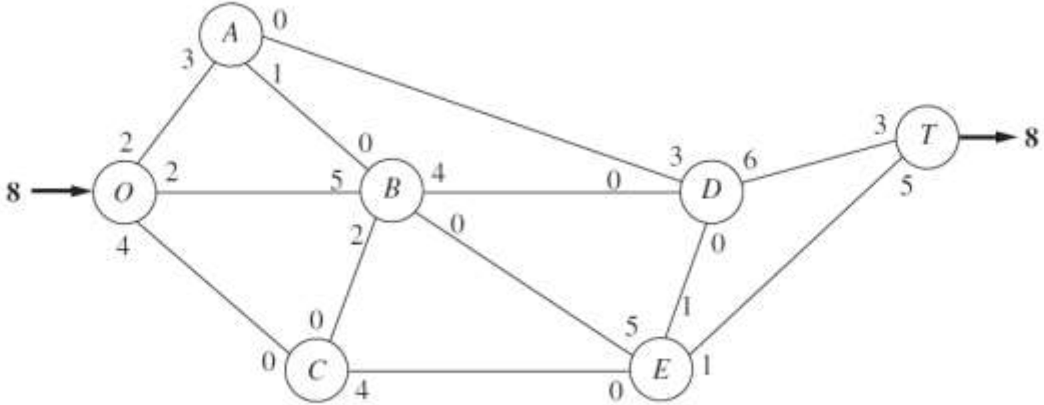
We now go to the augmenting path

$O \to A \to D \to T$

where the smallest capacity is 3, and move it

The capacity of link AD is now exhausted





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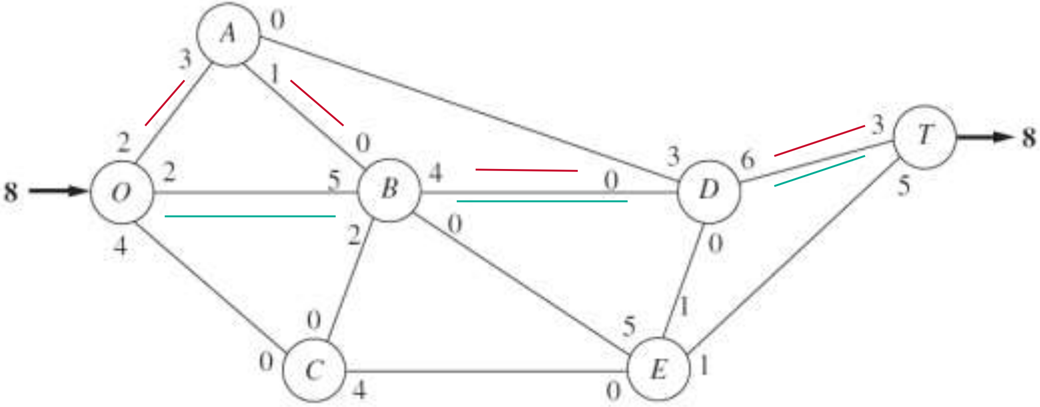
Assign a flow of 1 to the augmenting path

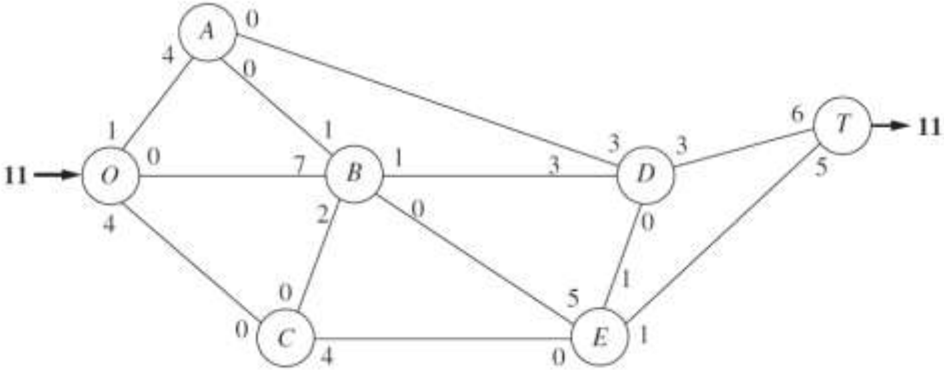
$O \rightarrow A \rightarrow B \rightarrow D \rightarrow T$

Assign a flow of 2 to the augmenting path

$O \rightarrow B \rightarrow D \rightarrow T$

The capacity of links AB and OB are now exhausted



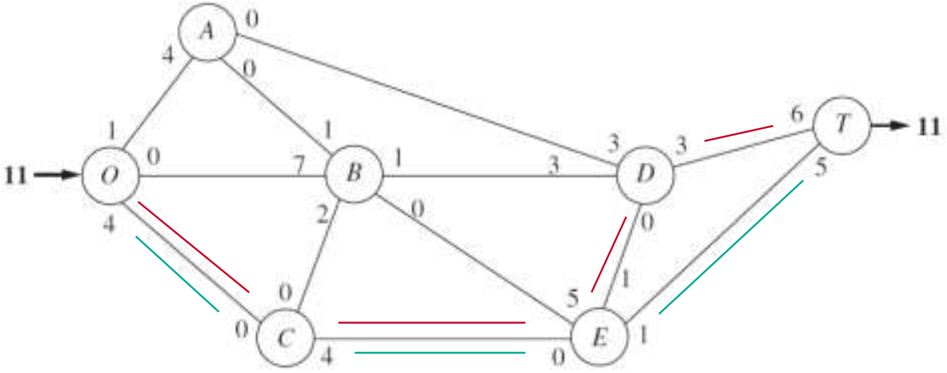


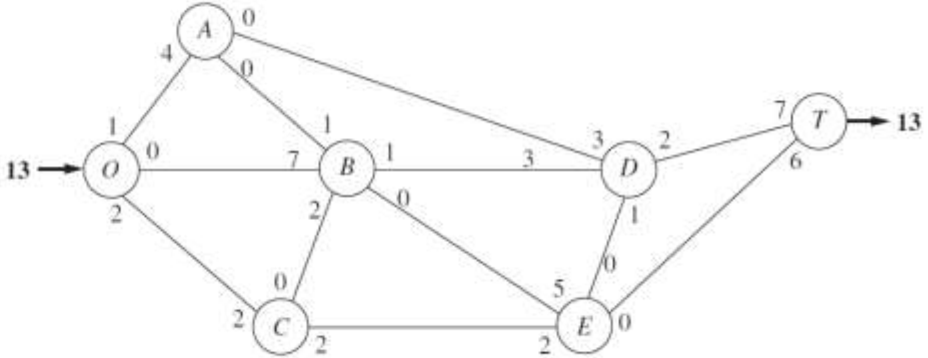
Assign a flow of 1 to the augmenting path

$O \rightarrow C \rightarrow E \rightarrow D \rightarrow T$

Assign a flow of 1 to the augmenting path

$O \rightarrow C \rightarrow E \rightarrow T$



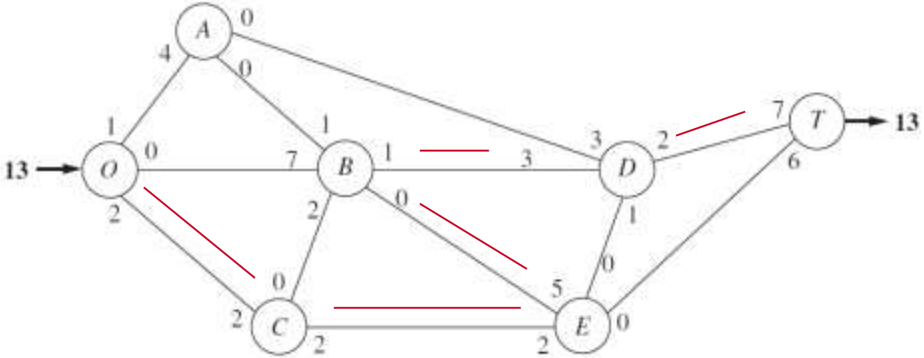


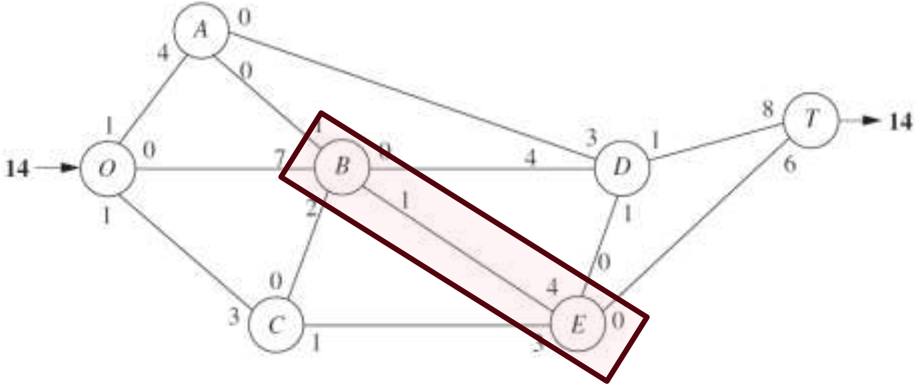
Assign a flow of 1 to the augmenting path

$O \rightarrow C \rightarrow E \rightarrow B \rightarrow D \rightarrow T$

The capacity of link BD is now exhausted

Anything weird here?





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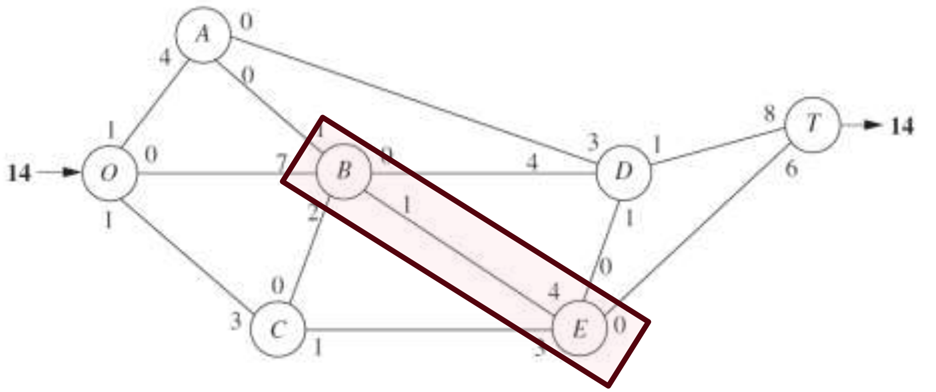
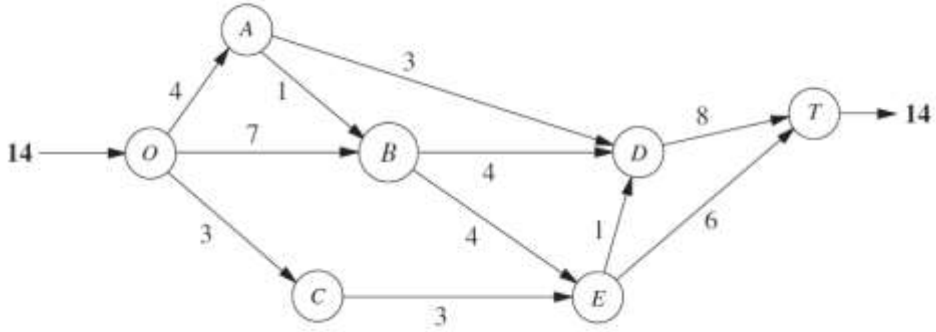
We have moved 'counter-current' – this is the same as reversing part of a previous flow

This was also the final move





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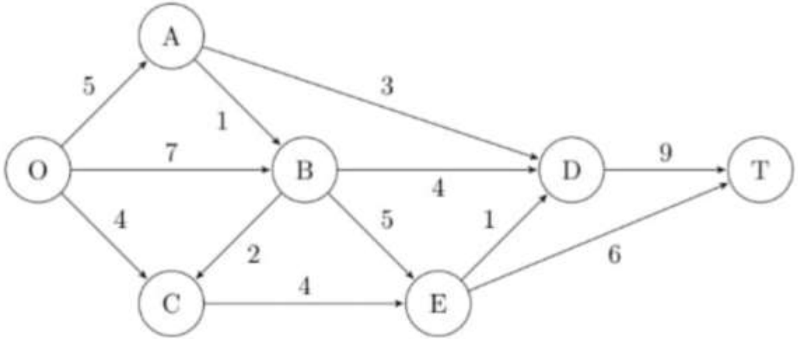
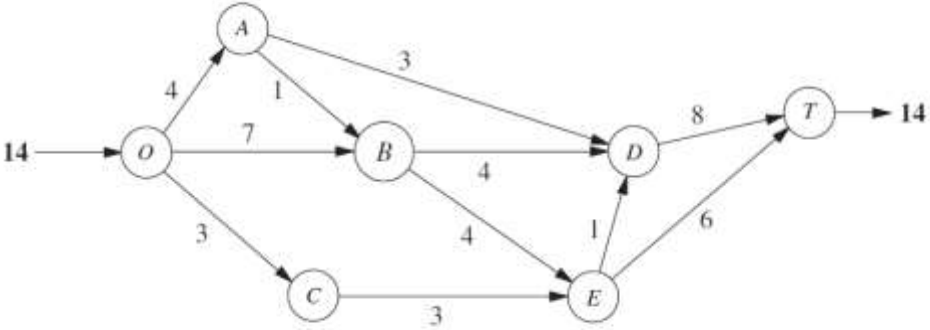
Check for yourself that

- No capacity has been violated
- No accumulation takes place at any node



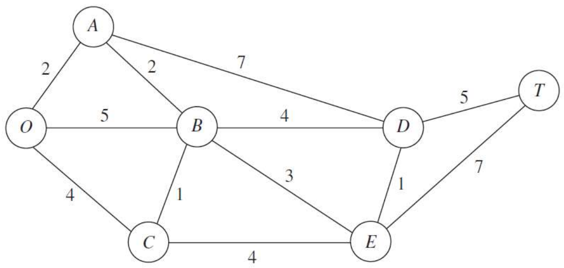


Three practical problems

- **Shortest path** from entrance O to scenic point T
- Minimum length of telephone lines covering all tracks (**minimum spanning tree**)

Solved

- **Maximum flow** of mini-trains carrying non trekkers from entrance O to scenic point T



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Source: https://www.klook.com/en-US/activity/28218-yosemite-park-giant-sequoia-day-tour-san-francisco/?

Notes from previous lessons

(1) Combinations, permutations, ~~variations;~~
(2) Graphical optimization exercise: it was minimize not maximize (teacher's error)

Combination and permutation with repetition ⋯ keep these formulae at hand

	Permutations n elements in classes of k (~~variations~~)	Combinations n elements in classes of k
No repetition	$\frac{n!}{(n-k)!}$	$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$
Repetition	n^k	$\binom{n+k-1}{k}$

Somewhere you see $n!$ called a permutation and $\frac{n!}{(n-k)!}$ a variation;

MOST WORKS call these latter permutation in classes of k

Combinations and permutations with repetition: example three objects ABC in groups of 2

	Permutations 3 elements in classes of 2 (variations)	Combinations 3 elements in classes of 2
No repetition	AB,BA,AC,CA,BC,CB (3!/1!=6)	AB,AC,BC (3!/2!=3)
Repetition	AA,BB,CC,AB,BA,AC, CA,BC,CB (3^2=9)	AA,BB,CC,AB,AC,BC (4!/(2!2!)=6)

Consider the following model (Hillier exercise 3.4-7): Minimize ← I had written Maximize My mistake

$$Z = 40x_1 + 50x_2$$

subject to

$$2x_1 + 3x_2 \geq 30$$
$$x_1 + x_2 \geq 12$$
$$2x_1 + x_2 \geq 20$$

and

$$x_1 \geq 0$$
$$x_2 \geq 0$$

Use the graphical method to solve this model.

Drawing the graph (next page)it becomes evident that the solution is given by the intersection of

$$2x_1 + x_2 = 20$$

With

$$2x_1 + 3x_2 = 30$$

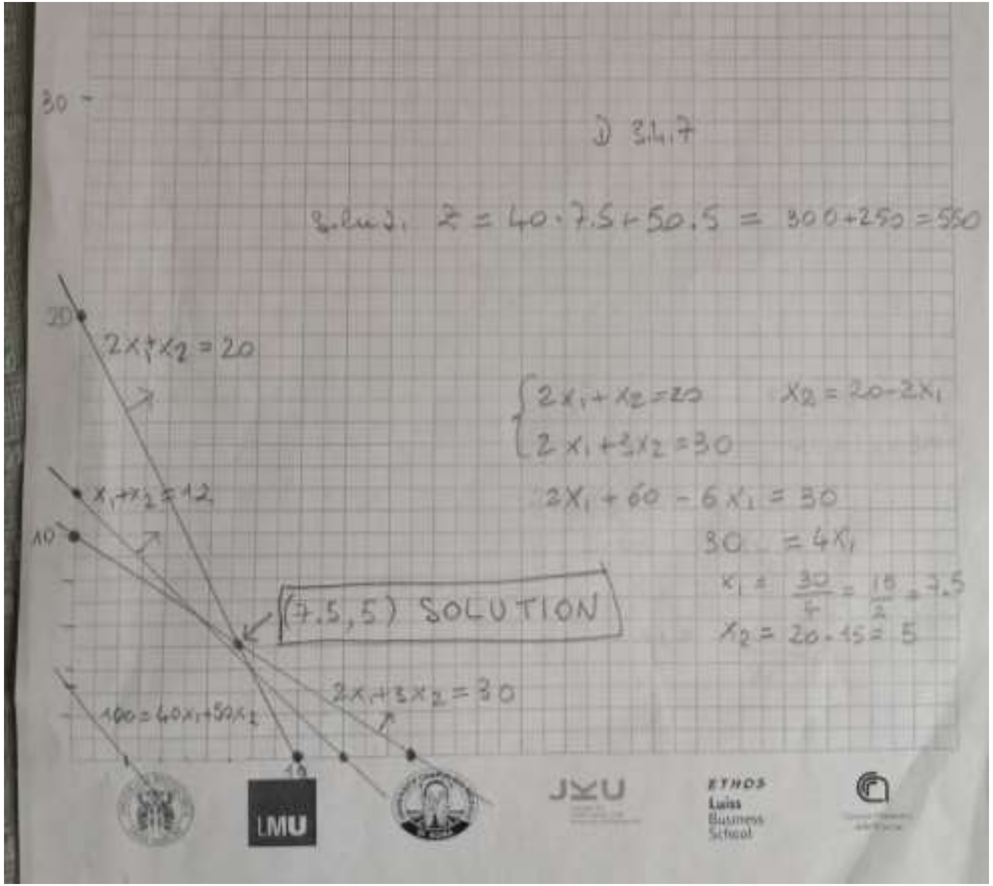
The first equation gives $x_2 \geq 20 - 2x_1$ which plugged into $2x_1 + 3x_2 \geq 30$ gives

$$2x_1 + 3(20 - 2x_1) = 30$$
$$-4x_1 + 60 = 30$$
$$x_1 = 30/4 = 7.5$$

That plugged into $2x_1 + x_2 = 20$ gives

$$15 + x_2 = 20$$
$$x_2 = 5$$

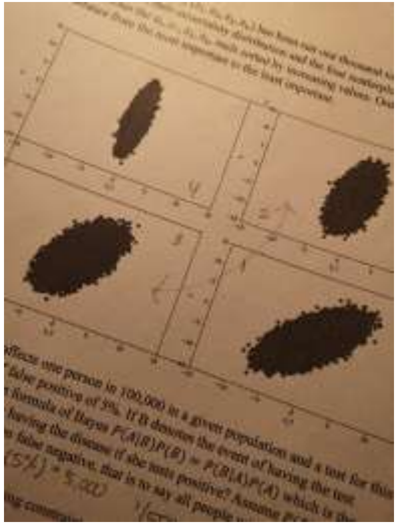
So $(x_1, x_2) = (7.5, 5)$ is the solution that plugged into Z give $40*7.5 + 50*5 = 550$


30
20
10
2x1+x2=20
x1+x2=12
100=40x1+50x2
2x1+3x2=30
(7.5,5) SOLUTION
Z = 40·7.5+50·5 = 300+250 = 550
2x1+x2=20
2x1+3x2=30
X2 = 20-2X1
2X1+60-6X1=30
30 = 4X1
x1 = 30/4 = 15/2 = 7.5
X2 = 20-15 = 5
LMU
JKU
ETHOS
Luiss
Business
School

Notes from midterm assignment

Bayes, binomial, decision variables

Rule



Please write the solution here

… not on the assignment form

Binomial distribution

Question 4) Launching two coins 5 times which is the probability of getting two heads exactly twice.

Which is that is done each of the five times?

➔Two coins are launched;

What are the possible outcomes of this single trial? imagine two different coins, e.g. of different colour!

HH	HT
TH	TT

In how many ways five experiments can generate 'success'=2 heads twice and failure=anything else three times?

➔The success could appear at 12,13,14,15,23,24,25,34,35,45 (read: at the first and second experiment, at the first and third experiment, … at the fourth and fifth experiment). These are 10 combinations of five experiments in classes of two, equal to $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5*4}{2} = 10$

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Which is the probability of HH (p=1/4) appearing twice

➔ $\left(\frac{1}{4}\right)^2$

Which is the probability of non-HH (p=3/4) appearing three times

➔ $\left(\frac{3}{4}\right)^3$

➔ Hence P= $\binom{5}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 = 10\frac{1}{16}\frac{27}{64} = \frac{270}{1024} \sim 0.26$

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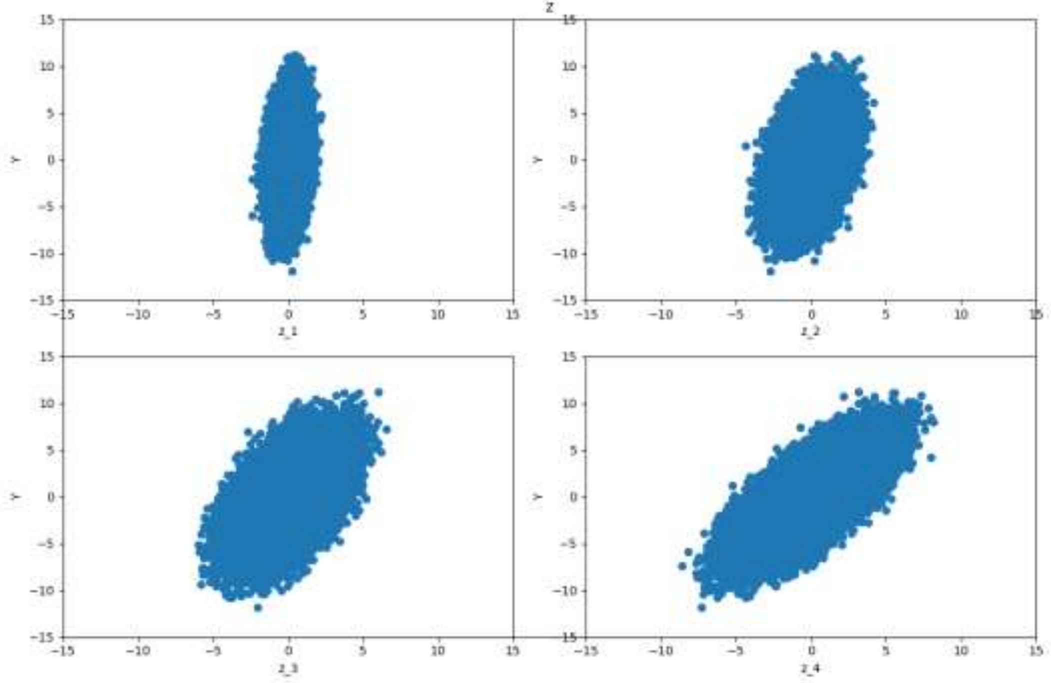
Scatterplots

Question 1) A function $y = f(z_1, z_2, z_3, z_4)$ has been run one thousand times sampling z_1, z_3, z_3, z_4 from their uncertainty distribution and the four scatterplots show y on the ordinate versus the z_1, z_2, z_3, z_4, each sorted by increasing values. Order z_1, z_2, z_3, z_4 by importance from the most important to the least important.

Clue: it is not only a matter of pattern, but of pattern driven by the dependent variable in the abscissa! The first plot (y versus z_1) has pattern but it is not driven by z_1



Bayes

Question 2) Disease A affects one person in 100,000 in a given population and a test for this disease has a rate of false positive of 5%. If B denotes the event of having the test positive and using the formula of Bayes P(A | B)P(B)=P(B | A)P(A) which is the probability of a person having the disease if she tests positive? Assume P(B | A)=1, meaning that there are no false negative, that is to say all people with disease A test positive.

P(A)=0.00001
P(B)=0.00001+ 0.05~0.05
P(B | A)=1

Hence $P(A|B)P(B) = P(B|A)P(A) \Rightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{1*0.00001}{0.05} = 0.0002 = 0.02\%$

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Linear Programming

Question 6) From Hillier, exercise 3.4-11*. The Medequip Company produces precision medical diagnostic equipment at two factories. Three medical centers have placed orders for this month's production output. The table below shows what the cost would be for shipping each unit from each factory to each of these customers. Also shown are the number of units that will be produced at each factory and the number of units ordered by each customer.

From \ To	Unit Shipping Cost			Output
	Customer 1	Customer 2	Customer 3	
Factory 1	$600	$800	$700	400 units
Factory 2	$400	$900	$600	500 units
Order size	300 units	200 units	400 units	

A decision now needs to be made about the shipping plan for how many units to ship from each factory to each customer.

(a) Formulate a linear programming model for this problem (without solving it).
(b) Knowing that the solution is

From Factory 1, ship 200 units to Customer 2 and 200 units to Customer 3.
From Factory 2, ship 300 units to Customer 1 and 200 units to Customer 3.

Check that this solution satisfies the constraints you have written and compute the value of the objective function.

Linear Programming

Maximize or minimize?

➔Minimize

What?

➔Cost

➔Hence your Z must include these numbers!

From \ To	Unit Shipping Cost: Customer 1	Customer 2	Customer 3	Output
Factory 1	$600	$800	$700	400 units
Factory 2	$400	$900	$600	500 units
Order size	300 units	200 units	400 units	

Not those

…nor those

➔So however you number the decision variable …

➔… for example x_1 to x_6 row-wise, the Z must include them plus the costs!

From \ To	Unit Shipping Cost			Output
	Customer 1	Customer 2	Customer 3	
Factory 1	\$600 x_1	\$800 x_2	\$700 x_3	400 units
Factory 2	\$400 x_4	\$900 x_5	\$600 x_6	500 units
Order size	300 units	200 units	400 units	

➔Minimize Z= $600x_1 + 800x_2 + 700x_3 + 400x_4 + 900x_5 + 600x_6$

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From \ To	Unit Shipping Cost			Output
	Customer 1	Customer 2	Customer 3	
Factory 1	\$600 x_1	\$800 x_2	\$700 x_3	400 units
Factory 2	\$400 x_4	\$900 x_5	\$600 x_6	500 units
Order size	300 units	200 units	400 units	

While these numbers are constraints

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15.

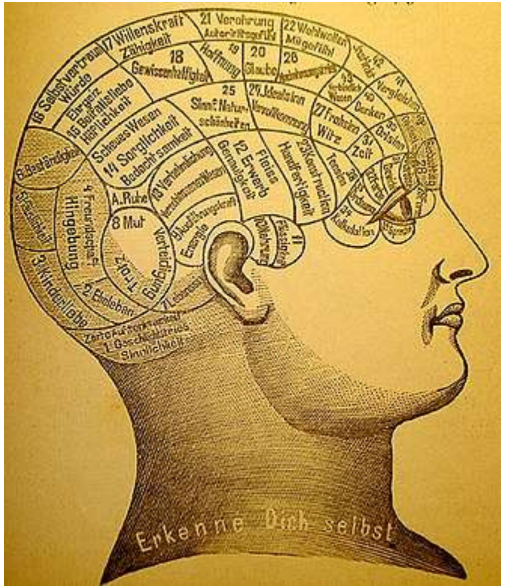
Integer Programming

Intuitions and fallacies. Why is it more difficult than LP. Integer and binary problems. Examples. Solution via branch and bound. Take home points. Hillier 2014, chapter 12.

Integer programming; intuition and fallacies

If the solutions need to be integer, there will be less of them, so Integer Programming (**IP**) will be easier than Linear Programming (**LP**)

- Yes, there will be less solutions, but still a very large numbers if they have to be found ‘by inspection’
- The simplex solution of an IP treated as if it were an LP (what is called LP relaxation) generally generate unfeasible solutions



A phrenological mapping of the brain. Source: Wikipedia Commons

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Moving from LP to IP which of the four assumptions of LP will need to fall?

Proportionality: The contribution of each activity to the value of the objective function Z is proportional to the level of the activity x_j increase in Z that , as represented by the $c_j x_j$ term in the objective function
Additivity: Every function in a linear programming model (whether the objective function or the function on the left-hand side of a functional constraint) is the sum of the individual contributions of the respective activities
Divisibility: Decision variables in a linear programming model are allowed to have any values, including noninteger values, that satisfy the functional and nonnegativity constraints. Thus, these variables are not restricted to just integer values. Since each decision variable represents the level of some activity, it is being assumed that the activities can be run at fractional levels
Certainty: The value assigned to the parameters (the a_j^i's, b_i's, and c_j's) of a linear programming model are assumed to be known constants

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YES, NO decision variables

An important class of IP involves binary decision variables that can be represented as (0,1)

$$x_j = \begin{cases} 1 \text{ if decision} = \text{yes} \\ 0 \text{ if decision} = \text{no} \end{cases}$$

When this is the case the IP problem is said to be a Binary Integer Programming (BIP) problem

A prototype example: building or not building?

■ **TABLE 12.1** Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

$$x_1 = \begin{cases} 1 \text{ if decision} = \text{yes build a factory in Los Angeles} \\ 0 \text{ if decision} = \text{no, don't build a factory in Los Angeles} \end{cases}$$

The choice is if building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building **at most one** new warehouse, but the choice of location is restricted to a city where a new factory is being built.

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The choice is if building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building at most one new warehouse, but the choice of location is restricted to a city where a new factory is being built.

➔ x_1 and x_2 can both be 1, but x_2 and x_3 will depend upon the choice made for x_1, x_2

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$$x_1 = \begin{cases} 1 \text{ if decision} = \text{yes build a factory in Los Angeles} \\ 0 \text{ if decision} = \text{no, don't build a factory in Los Angeles} \end{cases}$$

It is easy to see that the function to be maximized is

$$Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

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$$x_1 = \begin{cases} 1 \text{ if decision} = \text{yes build a factory in Los Angeles} \\ 0 \text{ if decision} = \text{no, don't build a factory in Los Angeles} \end{cases}$$

And an evident constraint is

$$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$$

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1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

$$x_1 = \begin{cases} 1 \text{ if decision} = \text{yes build a factory in Los Angeles} \\ 0 \text{ if decision} = \text{no, don't build a factory in Los Angeles} \end{cases}$$

Note: $x_3 = \text{yes}$ only if $x_1 = \text{yes}$

Likewise: $x_4 = \text{yes}$ only if $x_2 = \text{yes}$

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TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

$x_3 = 1$ only if $x_1 = 1$
$x_4 = 1$ only if $x_2 = 1$

So, knowing that al variables need to be either 0 or 1 a possible way to include this contingency is the constraint

$$x_3 \leq x_1$$
$$x_4 \leq x_2$$

TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
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4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

So, knowing that al variables need to be either 0 or 1 a possible way to include this contingency is the constraint

$$x_3 \leq x_1$$
$$x_4 \leq x_2$$

Since we only want at most one warehouse, it should also be
$x_3 + x_4 \leq 1$

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TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

Wrapping up, here the BIP problem:

$x_3 \leq x_1$
$x_4 \leq x_2$
rewritten in standard form →

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$

Subject to:

$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$

$-x_1 + x_3 \leq 0$

$-x_2 + x_4 \leq 0$

$x_3 + x_4 \leq 1$

and

$x_j \leq 1$

$x_j \geq 0$

x_j integer for $j = 1,2,3,4$

or

x_j binary for $j = 1,2,3,4$

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How many problems can be framed as BIP?

Investment decisions

Each yes-or-no decision:

Should we make a certain fixed investment?

Decision variable $x_j = \begin{cases} 1 & \text{if yes} \\ 0 & \text{if no} \end{cases}$

Siting decision

Each yes-or-no decision:

Should a certain site be selected to build a facility?

Decision variable $x_j = \begin{cases} 1 & \text{if yes} \\ 0 & \text{if no} \end{cases}$

How many problems can be framed as BIP?

Relocating/restructuring, etc.?

- Each yes-or-no decision:
- Should a certain plant remain open?
- Should a certain site be selected for a new plant?
- Should a certain distribution center remain open?
- Should a certain site be selected for a new distribution center?

How many problems can be framed as BIP?

Source: Wikipedia Commons

Dispatching decisions

Each yes-or-no decision:
Should a certain route be selected for one of the trucks?

Decision variable $x_j = \begin{cases} 1 & \text{if yes} \\ 0 & \text{if no} \end{cases}$

Or in more complicated arrangements: Should all the following be selected simultaneously for a delivery run:

1. A certain route,
2. A certain size of truck, and
3. A certain time period for the departure?

Decision variable $x_j = \begin{cases} 1 & \text{if yes} \\ 0 & \text{if no} \end{cases}$

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How many problems can be framed as BIP?

An airline application: Assigning crews to sequences of flights (crew scheduling problem). In a previous step of the analysis 12 crew flight sequences (ordered from one to a max of five), and the problem is to choose three of them so that all flights would be covered

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

Z is easy: If x_j =(0,1) decides if assigning the sequence to one of the three crews, then we must minimize:

$$Z = 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$$

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

Since the crews are three it must be

$$\sum_{j=1}^{12} x_j = 3$$

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

Then for each of the 11 flights (1. San Francisco to Los Angeles all the way to 11. Seattle to Los Angeles) it must be that the sum of the coefficients covering that flight add up to one or more (more crews can fly on a flight – there can be a non working crew that still needs to be paid)

1. $x_1 + x_4 + x_7 + x_{10} \geq 1$
2. $x_2 + x_5 + x_8 + x_{11} \geq 1$

…

11. $x_6 + x_9 + x_{10} + x_{11} + x_{12} \geq 1$

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

So wrapping up the problem is:

Minimize $Z = 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$

Subject to

$\sum_{j=1}^{12} x_j = 3$ and the 11 constraints

$x_1 + x_4 + x_7 + x_{10} \geq 1$
$x_2 + x_5 + x_8 + x_{11} \geq 1$
…
$x_6 + x_9 + x_{10} + x_{11} + x_{12} \geq 1$

Are we done?

x_j binary for $j = 1,2,\ldots 12$

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

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Minimize

$$Z = 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$$

Verify that one optimal solution for this BIP model is

$x_3 = 1$ (assign sequence 3 to a crew)

$x_4 = 1$ (assign sequence 4 to a crew)

$x_{11} = 1$ (assign sequence 11 to a crew)

and all other $x_j = 0$

and that another optimal solution is

$x_1 = 1$

$x_5 = 1$

$x_{12} = 1$

and all other $x_j = 0$

And compute Z for the two options

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

Minimize

$$Z = 2x_1 + 3x_2 + 4x_3 + 6x_4 + 7x_5 + 5x_6 + 7x_7 + 8x_8 + 9x_9 + 9x_{10} + 8x_{11} + 9x_{12}$$

Verify that one optimal solution for this BIP model is
$x_3 = 1$ (assign sequence 3 to a crew)
$x_4 = 1$ (assign sequence 4 to a crew)
$x_{11} = 1$ (assign sequence 11 to a crew)
and all other x_j =0

and that another optimal solution is

$x_1 = 1$
$x_5 = 1$
$x_{12} = 1$
and all other x_j =0

And compute Z for the two options

Z=18

TABLE 12.4 Data for Example 3 (the Southwestern Airways problem)

	Feasible Sequence of Flights											
Flight	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1. San Francisco to Los Angeles	1			1			1			1		
2. San Francisco to Denver		1			1			1			1	
3. San Francisco to Seattle			1			1			1			1
4. Los Angeles to Chicago				2			2		3	2		3
5. Los Angeles to San Francisco	2					3				5	5	
6. Chicago to Denver				3	3				4			
7. Chicago to Seattle							3	3		3	3	4
8. Denver to San Francisco		2		4	4				5			
9. Denver to Chicago					2			2			2	
10. Seattle to San Francisco			2				4	4				5
11. Seattle to Los Angeles						2			2	4	4	2
Cost, $1,000's	2	3	4	6	7	5	7	8	9	9	8	9

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We just solved a **set covering problem**, (all flights need to be covered)

A related BIP is the **set partitioning problem**, where instead of e.g.

$$x_1 + x_4 + x_7 + x_{10} \geq 1$$

(previous problem) one would ask:

$$x_1 + x_4 + x_7 + x_{10} = 1$$

This would prevent more than one crew flying on the same flight

Source: https://airportwingspvtltd.wordpress.com/2016/01/04/role-and-responsibilities-of-cabin-crew/

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As mentioned, IP are in general more difficult than LP; though there are less solutions, there are many of them; e.g. for a BIP with ten decision variables the number of possible solutions is $\mathbf{2^{10} = 1{,}024}$

Why?

Permutations with repetition of ten elements in groups of 10

It is not forbidden to try a LP approach for a IP problem (**LP relaxation**), though in general there is no guarantee that the solution will be feasible for the IP

It is not forbidden to try a LP approach for a IP problem (**LP relaxation**), though in general there is no guarantee that the solution will be feasible for the IP

… but when the LP relaxation solution satisfies the integer restriction of the IP problem, this solution must be optimal for the IP problem as well (=the best among all LP solutions is also the best for the subset of the IP solutions)

The LP relaxation value for the optimization function Z is in any case an upper bound for the Z of the integer problem

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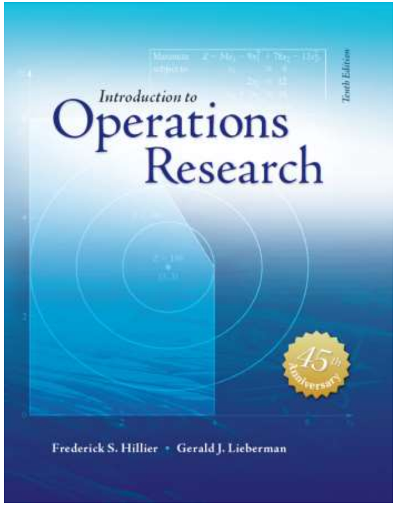
It is not forbidden to try a LP approach for a IP problem (**LP relaxation**), though in general there is no guarantee that the solution will be feasible for the IP

“Therefore, it is common for an IP algorithm to begin by applying the simplex method to the LP relaxation to check whether this fortuitous outcome has occurred”



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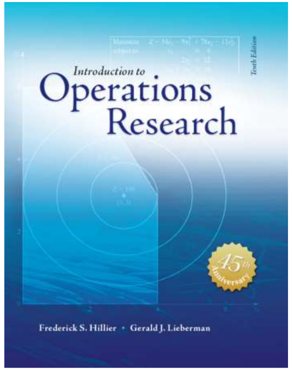
Exercise: (Hillier 12.1-2) A young couple, Eve and Steven, want to divide their main household chores (marketing, cooking, dishwashing, and laundering) between them so that each has two tasks but the total time they spend on household duties is kept to a minimum. Their efficiencies on these tasks differ, where the time each would need to perform the task is given by the following table:

	Time Needed per Week			
	Marketing	**Cooking**	**Dishwashing**	**Laundry**
Eve	4.5 hours	7.8 hours	3.6 hours	2.9 hours
Steven	4.9 hours	7.2 hours	4.3 hours	3.1 hours

- Write this as a binary integer programming problem
- Guess a solution



More tricks with binary variables. From Hillier, example pages 489−491

When one of two constraints must hold, for example

$$3x_1 + 5x_2 - 7x_3 \le 12$$

or

$$4x_1 + 2x_2 + x_3 \le 15$$

But not both we can use an auxiliary binary variable y and impose

$$3x_1 + 5x_2 - 7x_3 \le 12 + My$$
$$4x_1 + 2x_2 + x_3 \le 15 + M(1 - y)$$
$$x_i \ge 0$$
$$y \text{ binary}$$

Where M is the usual large number.
If $y = 0$ the first constraint holds, if $y = 1$ the second

"It is common for an IP algorithm to begin by applying the simplex method to the LP relaxation to check whether this fortuitous outcome has occurred"

This may or may not work see e.g. the simple example

Maximize $Z = x_2$ subject to

$-x_1 + x_2 \leq \frac{1}{2}$

$x_1 + x_2 \leq \frac{7}{2}$

← Find graphically the linear solution of this problem

and

$x_1 \geq 0, x_2 \geq 0$

x_1, x_2 integers ← I.e. removing this constraint

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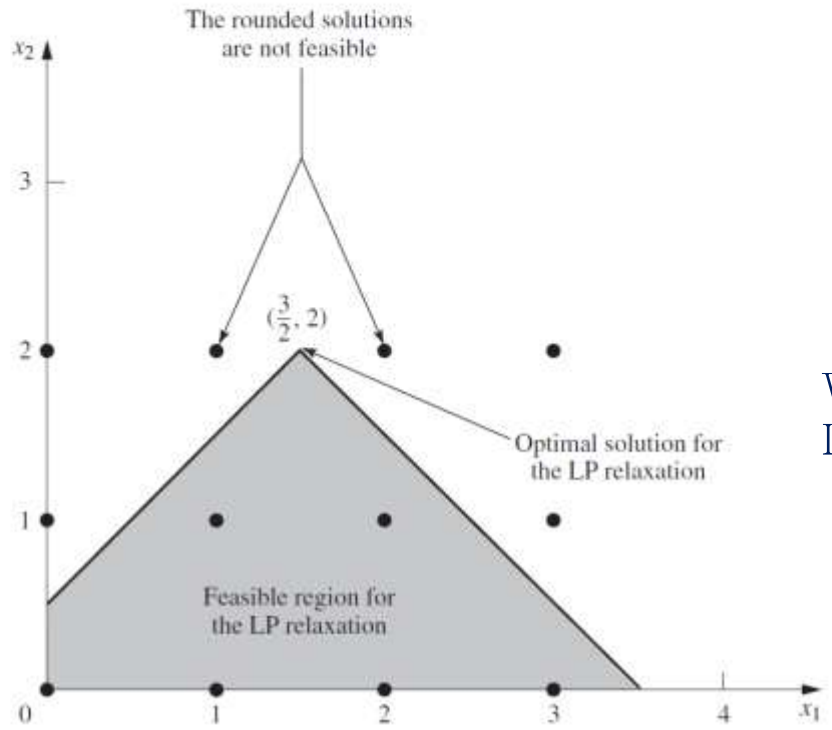

FIGURE 12.2
An example of an IP problem where the optimal solution for the LP relaxation cannot be rounded in any way that retains feasibility.

Which is instead the IP solutions?

Another case where the relaxation solution can be **not OK**

Maximize $Z = x_1 + 5x_2$ subject to

$x_1 + 10x_2 \leq 20$

$x_1 \leq 2$

and

$x_1 \geq 0, x_2 \geq 0$

x_1, x_2 integers

Find graphically the linear solution of this problem ⋯

⋯ i.e. removing this constraint

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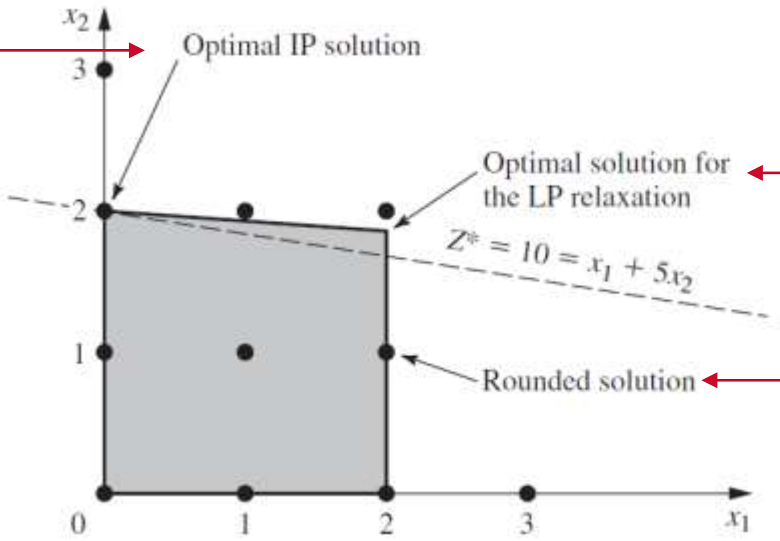
$x_1 = 0,$
$x_2 = 2$
$Z = 10$



$x_1 = 2,$
$x_2 = \frac{9}{5}$
$Z = 11$

$x_1 = 2,$
$x_2 = 1$
$Z = 7$

FIGURE 12.3
An example where rounding the optimal solution for the LP relaxation is far from optimal for the IP problem.

Did we violate the rule that the LP solution is an upper bound for the IP solution?

$$x_1 = 2, \quad x_2 = \frac{9}{5}, \quad Z = 11$$

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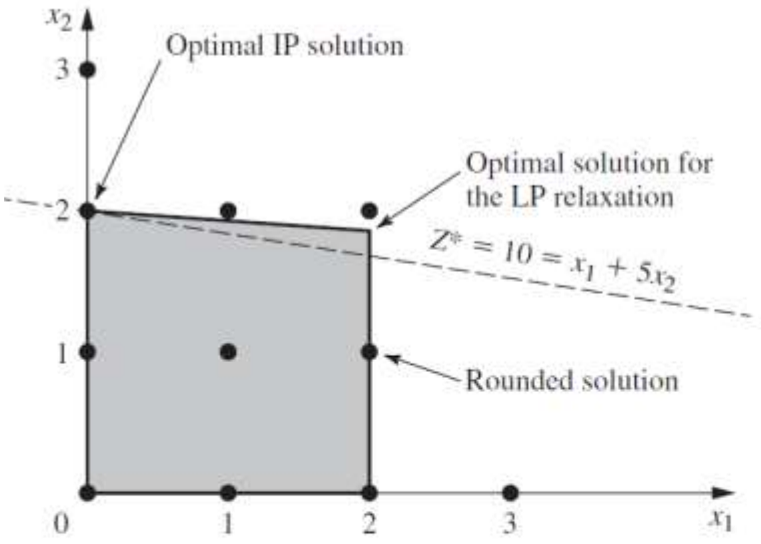
When there are may dimensions checking that the relaxation solution is OK can be tricky;

Here we have only 7 integer points in the feasible region, but the number of points grows exponentially with the number of dimensions



In many dimensions better use heuristic method (such as genetic algorithms, more later) that also work for nonlinear problems.

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But there are IP problems whose structure guarantees an integer solution; remember the Transportation Problem (Section 12);

The integer solutions property: For transportation problems where every supply s_i and demand d_i have an integer value, all basic feasible (BF) solutions (including an optimal one) also have integer values

TABLE 9.3 Constraint coefficients for P & T Co.

	Coefficient of:											
	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}
A = Cannery constraints	1	1	1	1								
					1	1	1	1				
									1	1	1	1
Warehouse constraints	1				1				1			
		1				1				1		
			1				1				1	
				1				1				1

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But there are IP problems whose structure guarantees an integer solution; remember from the section on Transportation Problem (Section 12);

Source: Wikipedia Commons

Other special cases are the assignment problem, the shortest-path problem, and the maximum flow problem

Charles Chaplin's Modern Times, source http://internationalcinemareview.blogspot.com/2013/04/charles-chaplin-modern-times.html

Source: https://www.yosemite.com/things-to-do/leisure-activities/valley-floor-tour/

Ramon Casas and Pere Romeu on a Tandem, Barcelona. Source: Wikipedia Commons

Level of difficulty of LP versus IP

	Difficulty of LP problem	Difficulty of IP problem
Source	Number of constraints	Number of integer variables
		Binary or general integer?
		Special form?

Source:
https://www.dreamstime.com/illustrati
on/accountant.html

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Back to out prototype example: building or not building?

■ **TABLE 12.1** Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

The choice is if building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building **at most one** new warehouse, but the choice of location is restricted to a city where a new factory is being built.

TABLE 12.1 Data for the California Manufacturing Co. example

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

If we apply LP relaxation replacing
x_j binary for $j = 1,2,3,4$
with
$x_j \geq 0$ for $j = 1,2,3,4$

We obtain $x_1, x_2, x_3, x_4 = \left(\frac{5}{6}, 1,0,1\right)$
with $Z = 16.5$

We round this to 16 and keep it as an upper bound for the IP problem

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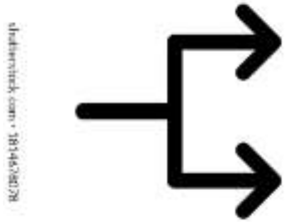
One method to solve IP problems: the branch-and-bound technique

- Branching (split the problem in two branches)
- Bounding (seek for a local optima for Z)
- Fathoming (Resolving the branching at fathomed the node)



Source: https://www.123rf.com/

Source: https://thesaurus.plus/synonyms/fathomed

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- Branching (split the problem in two branches)

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

$x_1 = 0$

Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \leq 10$
$x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

$x_1 = 1$

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

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- Branching (split the problem in two branches)

We are splitting following the order of the variables, i.e. here starting by x_1. Better strategies are available

- Branching (split the problem in two branches)

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and
x_j binary for $j = 1,2,3,4$

$x_1 = 0$

Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \leq 10$
$x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and
$x_j \geq 0$ for $j = 2,3,4$

$x_1 = 1$

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and
$x_j \geq 0$ for $j = 2,3,4$

The two subproblems are treated as linear instead of integer

- Bounding (seek for local optima for Z)

- Branching (split the problem in two branches)

Maximize $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-x_1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

x_j binary for $j = 1,2,3,4$

$x_1 = 0$:

Maximize $5x_2 + 6x_3 + 4x_4$
Subject to:
$3x_2 + 5x_3 + 2x_4 \leq 10$
$x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

$x_1 = 1$:

Maximize $Z = 9 + 5x_2 + 6x_3 + 4x_4$
Subject to:
$6 + 3x_2 + 5x_3 + 2x_4 \leq 10$
$-1 + x_3 \leq 0$
$-x_2 + x_4 \leq 0$
$x_3 + x_4 \leq 1$
and

$x_j \geq 0$ for $j = 2,3,4$

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Linear programming applied to these solutions yields

$x_1, x_2, x_3, x_4 = (0,1,0,1)$ with $Z = 9$

$x_1, x_2, x_3, x_4 = \left(1, \frac{4}{5}, 0, \frac{4}{5}\right)$ with $Z = 16.5$

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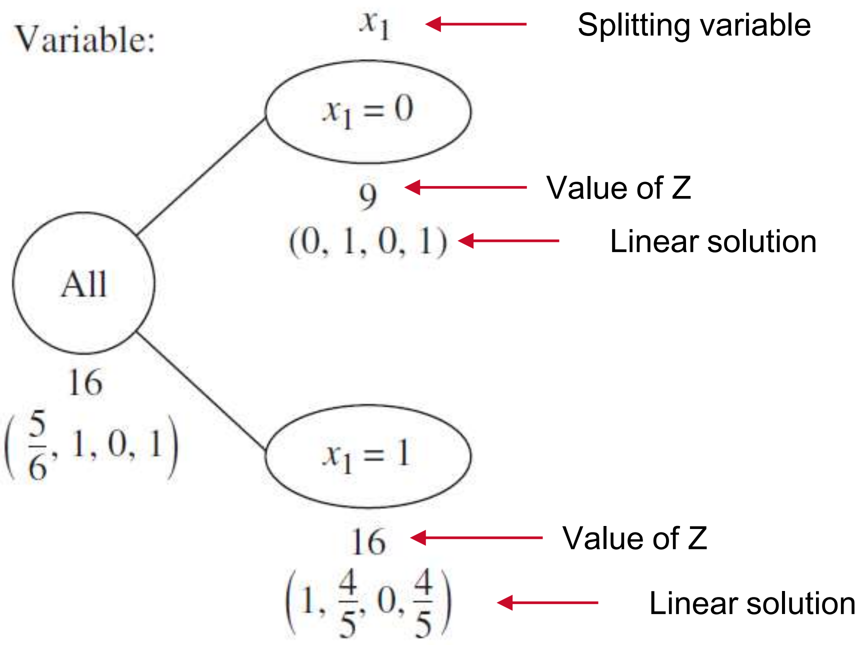
This is where we are at the end of the first bounding step:



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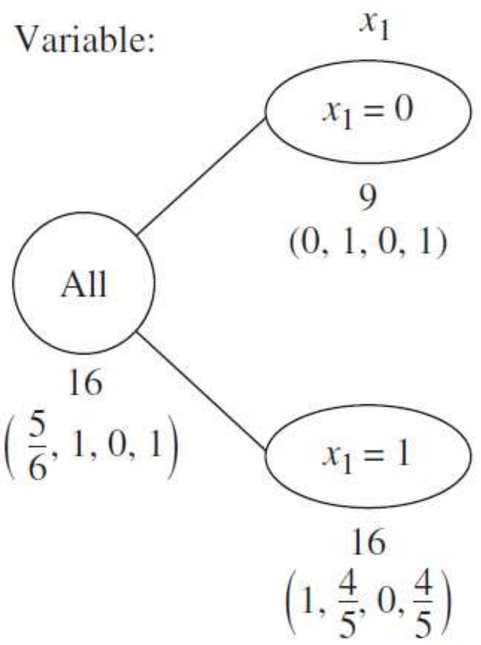
- Fathoming (Resolving the branching at fathomed the node)



Since it is optimal it does not pay to search for other solutions in this branch

This solution is made of integers! It is **hence optimal for the subproblem with $x_1 = 0$.** We call this now **the incumbent optimum** $Z^* = 9$ and say that the branch $x_1 = 0$ is fathomed; in the following we can get rid of all branches whose $Z \leq Z^* = 9$

This cannot be fathomed

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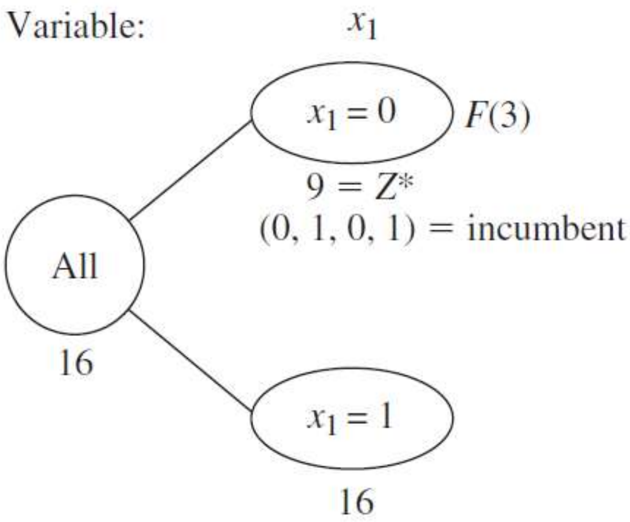
- Fathoming (Resolving the branching at fathomed the node)



In fact, there are 3 ways of fathoming:

Test 1: Its bound by being $\leq Z^*$

Test 2: Its LP relaxation has no feasible solutions

Test 3: The optimal solution for its LP relaxation is integer.

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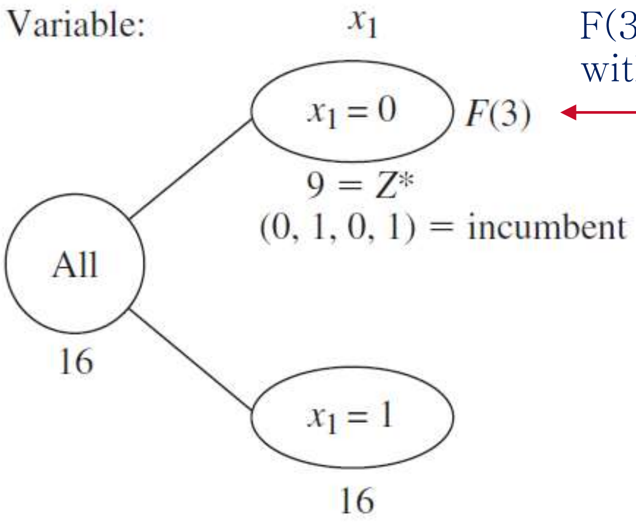
- Fathoming (Resolving the branching at fathomed the node)



F(3)=fathomed with test 3

In fact, there are 3 ways of fathoming:

Test 1: Its bound $\leq Z^*$

Test 2: Its LP relaxation has no feasible solutions

Test 3: The optimal solution for its LP relaxation is integer

If this solution is better than the incumbent, it becomes the new incumbent Z^*, and test 1 is reapplied to all previous unfathomed subproblems using this new larger Z^*

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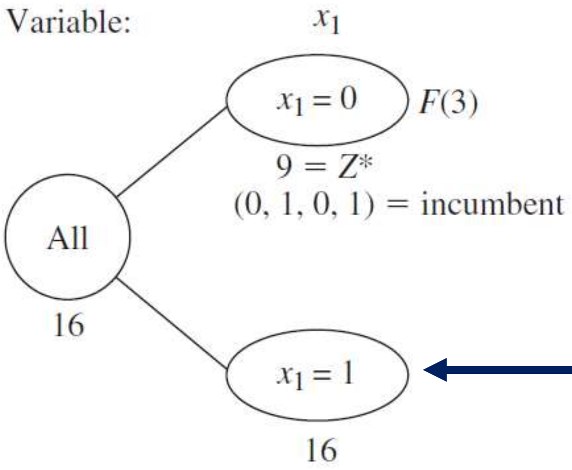
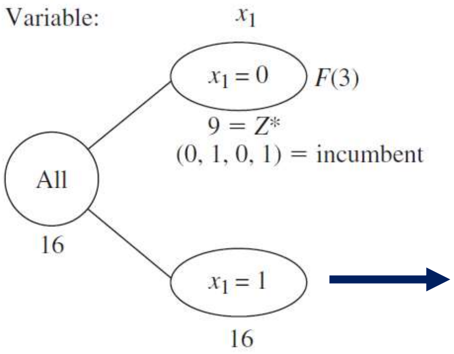
- Continuing the example



We now branch the $x_1 = 1$ problem by branching x_2 between 0 and 1

- Continuing the example



$x_2 = 0, x_1 = 1$

Maximize $Z = 9 + 6x_3 + 4x_4$

Subject to

Subject to:

$5x_3 + 2x_4 \leq 4$

$x_3 \leq 1$

$x_4 \leq 0$

$x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 3,4$

$x_2 = 1, x_1 = 1$

Maximize $Z = 9 + 5 + 6x_3 + 4x_4$

Subject to:

$5x_3 + 2x_4 \leq 1$

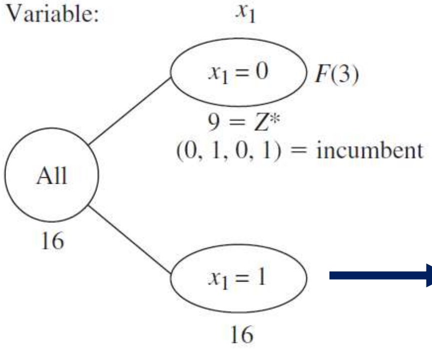
$x_3 \leq 1$

$x_4 \leq 1$

$x_3 + x_4 \leq 1$

$x_j \geq 0$ for $j = 3,4$

- Continuing the example



$x_2 = 0, x_1 = 1$
Maximize $Z = 9 + 6x_3 + 4x_4$
Subject to
Subject to:
$5x_3 + 2x_4 \leq 4$
$x_3 \leq 1$
$x_4 \leq 0$
$x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 3,4$

$x_2 = 1, x_1 = 1$
Maximize $Z = 9 + 5 + 6x_3 + 4x_4$
Subject to:
$5x_3 + 2x_4 \leq 1$
$x_3 \leq 1$
$x_4 \leq 1$
$x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 3,4$

Linear programming applied to these solutions yields

$x_1, x_2, x_3, x_4 = \left(1,0,\frac{4}{5},0\right)$ with $Z = 13.8$

$x_1, x_2, x_3, x_4 = \left(1,1,0,\frac{1}{2}\right)$ with $Z = 16$

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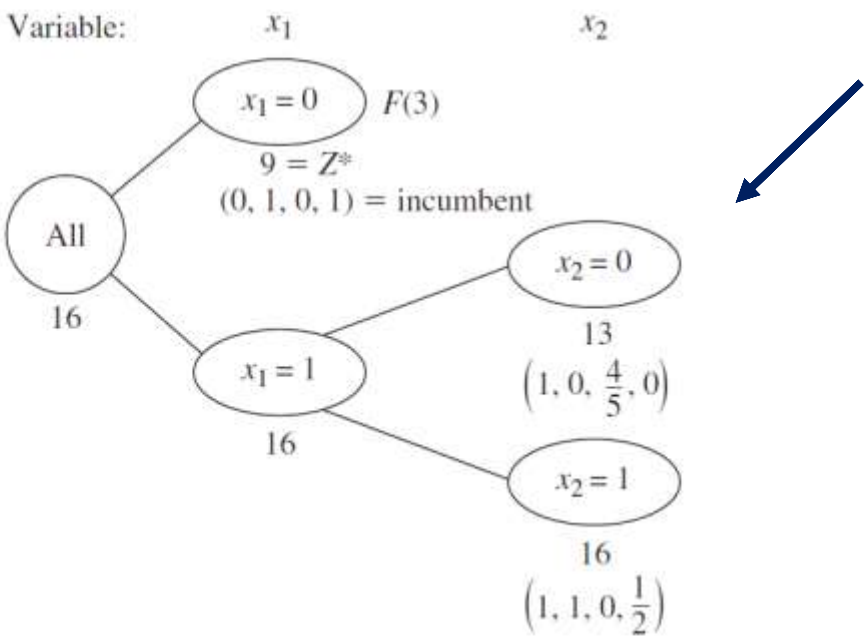
- Continuing the example



This is where we are now; no problem has been bound or fathomed at this step

Test 1: Its bound $\leq Z^*$ NO

Test 2: Its LP relaxation has no feasible solutions NO

Test 3: The optimal solution for its LP relaxation is integer NO

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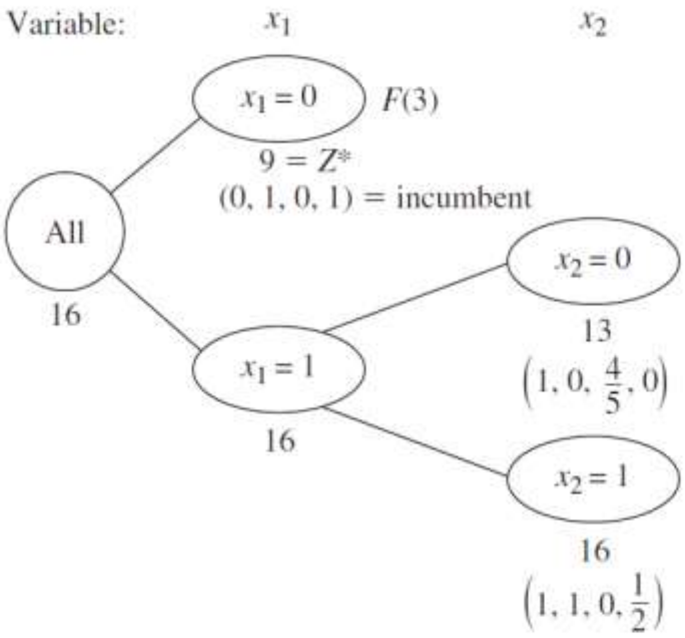
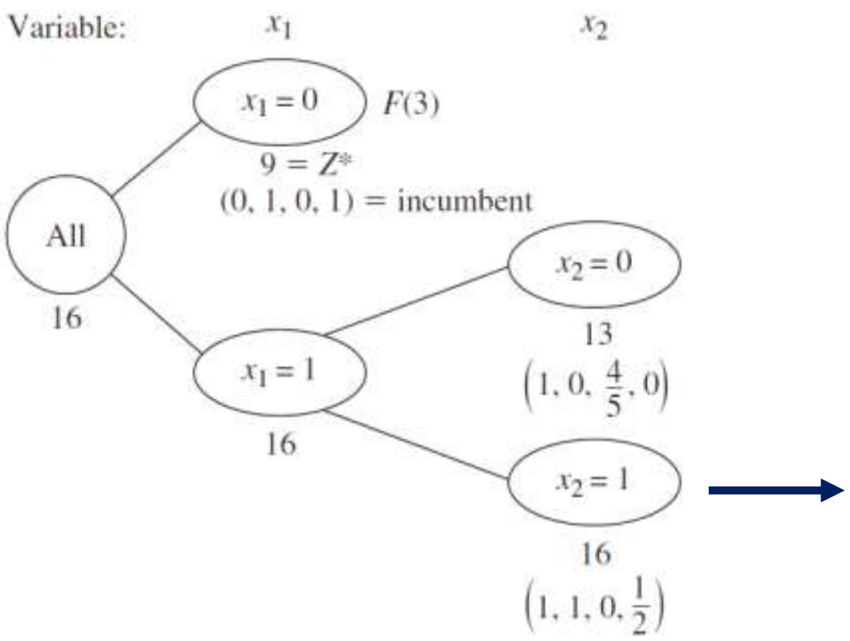
- Continuing the example



Since the problem $x_2 = 1$ has the larger Z we branch this solution splitting on x_3

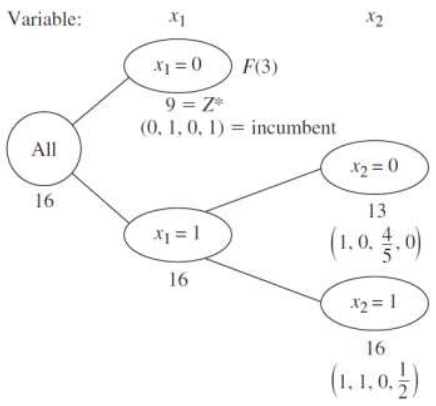
- Continuing the example; note how both Z and the constraints change to adopt to the new values



$x_3 = 0$, $x_1 = 1$, $x_2 = 1$
Maximize $Z = 14 + 4x_4$
Subject to:
$2x_4 \leq 1$ This was $5x_3 + 2x_4 \leq 1$
$x_4 \leq 1$
$x_4 \leq 1$ This was $x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 4$

$x_3 = 1$, $x_1 = 1$, $x_2 = 1$
Maximize $Z = 20 + 4x_4$
Subject to:
$2x_4 \leq -4$ This was $5x_3 + 2x_4 \leq 1$
$x_4 \leq 1$
$x_4 \leq 0$ This was $x_3 + x_4 \leq 1$
$x_j \geq 0$ for $j = 4$

- Continuing the example



$x_3 = 0, x_1 = 1,\ x_2 = 1$
Maximize $Z = 14 + 4x_4$
Subject to:
$2x_4 \leq 1$
$x_4 \leq 1$
$x_4 \leq 1$
$x_j \geq 0$ for $j = 4$

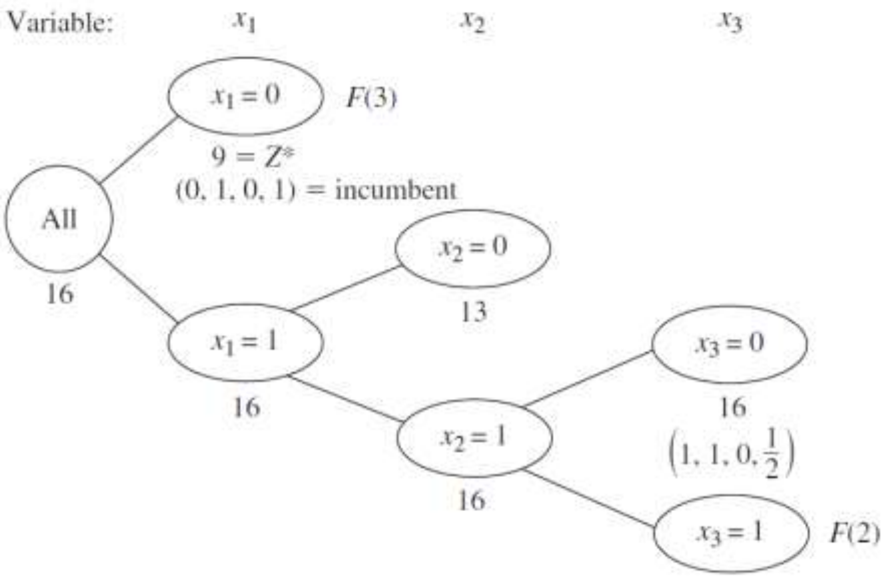
$x_3 = 1, x_1 = 1,\ x_2 = 1$
Maximize $Z = 20 + 4x_4$
Subject to:
$2x_4 \leq -4$
$x_4 \leq 1$
$x_4 \leq 0$
$x_j \geq 0$ for $j = 4$

Linear programming applied to these solutions yields no feasible integer solution

$x_1, x_2, x_3, x_4 = \left(1,1,0,\frac{1}{2}\right)$ with $Z = 16$

$x_1, x_2, x_3, x_4 =$ no feasible solution

- Continuing the example

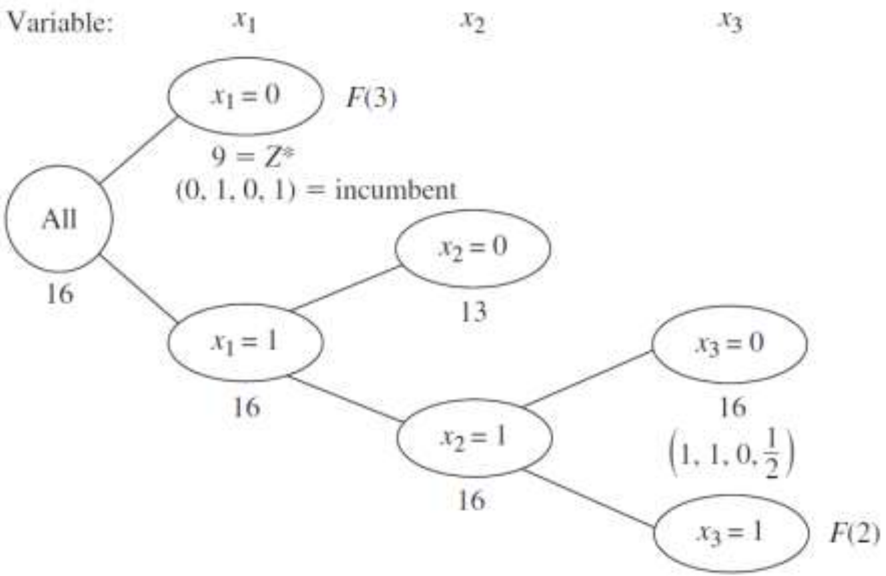


This is where we are now, with one solution fathomed and one open

No test failed

Test 2 failed

- Continuing the example)

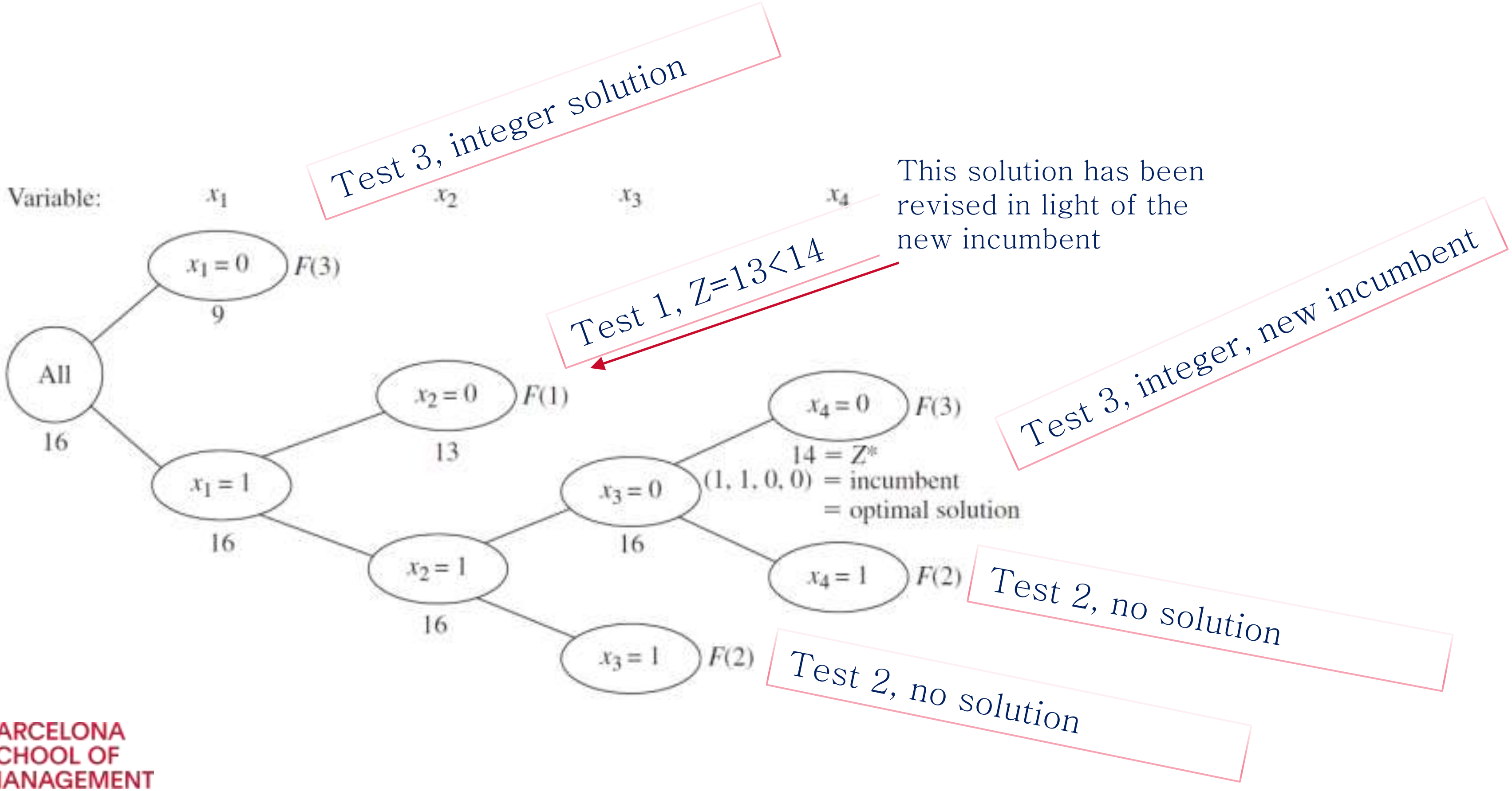


We now branch the problem with $x_3 = 0$, but since only variable x_4 is left fixing it generates directly a solution!

For $x_4 = 0$
$x_1, x_2, x_3, x_4 = (1,1,0,0)$ with $Z = 14$

For $x_4 = 1$
$x_1, x_2, x_3, x_4 = (1,1,0,1)$ unfeasible

- Continuing the example



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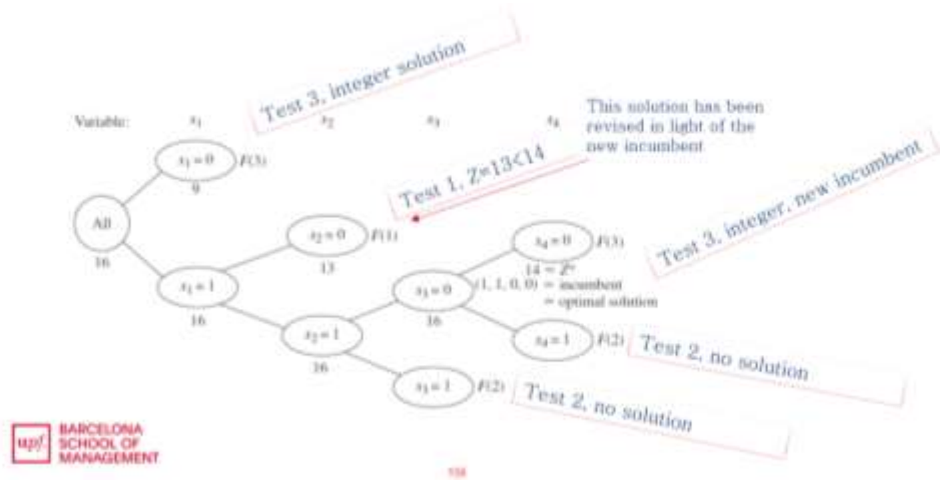
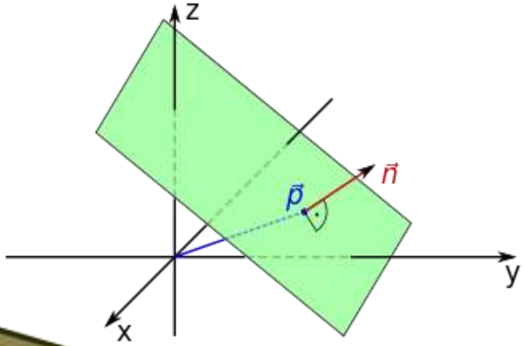
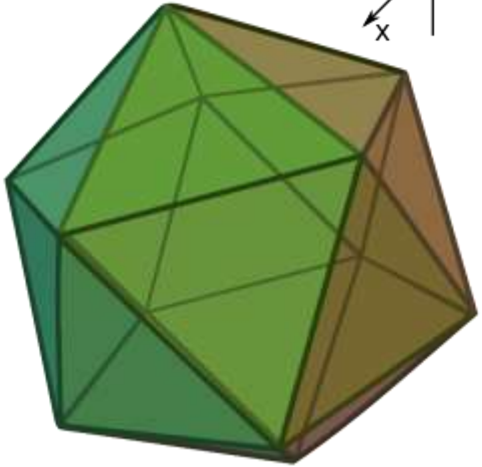


The solution is laborious, Needs book-keeping of how objective and constraints change in the various branches, and repeated recourse to LP, simplex calculations





Source (both images): Wikipedia Commons

Some take home points

Integer programming and linear programming: LP=convex polyhedron touched by the hyperplane of the objective function; the IP solutions instead are isolated point inside the polyhedron

Find these points may not be easy but the LP solution is an upper bound for the Z of IP

Panettone with raisins inside

Source: https://leitesculinaria.com/478/recipes-cranberry-pistachio-panettone.html

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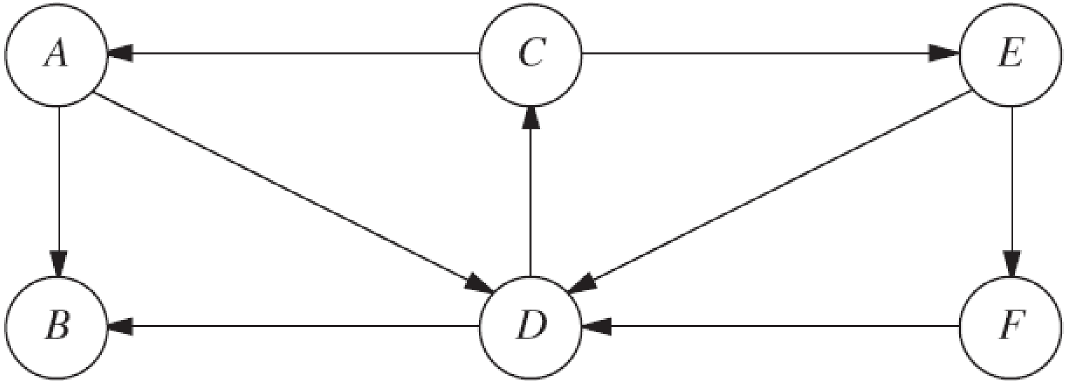
Homework

1) Consider the following directed network (Hillier 10.2-1)



(a) Find a directed path from node A to node F, and then identify three other undirected paths from node A to node F.

(b) Find three directed cycles. Then identify an undirected cycle that includes every node.

(c) Identify a set of arcs that forms a spanning tree.

(d) Use the process illustrated in Fig. 10.3 to grow a tree one arc at a time until a spanning tree has been formed. Then repeat this process to obtain another spanning tree. [Do not duplicate the spanning tree identified in part (c).]

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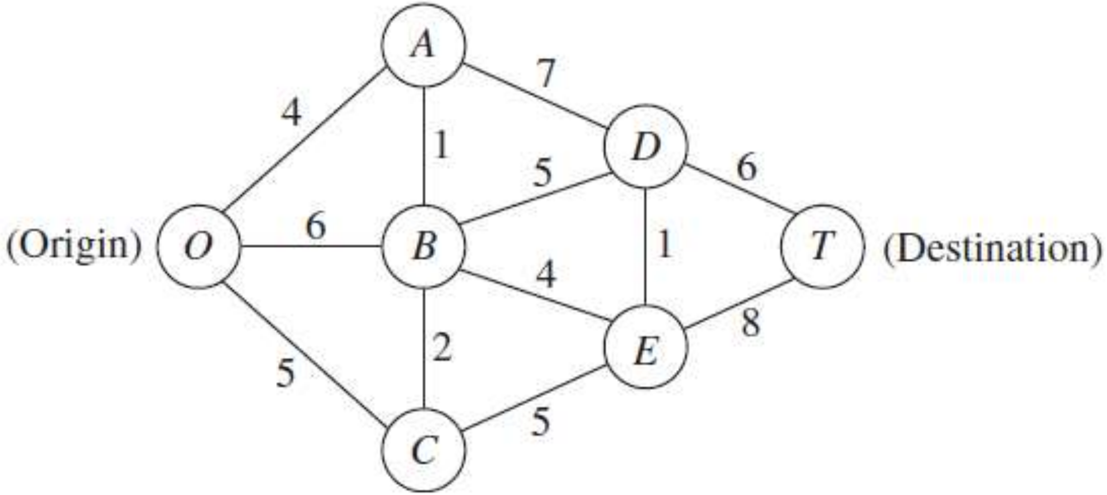
Homework 2) You need to take a trip by car to another town that you have never visited before. Therefore, you are studying a map to determine the shortest route to your destination. Depending on which route you choose, there are five other towns (call them A, B, C, D, E) that you might pass through on the way. The map shows the mileage along each road that directly connects two towns without any intervening towns. These numbers are summarized in the following table, where a dash indicates that there is no road directly connecting these two towns without going through any other towns. Formulate this problem as a shortest-path problem by drawing a network where nodes represent towns, links represent roads, and numbers indicate the length of each link in miles.

	Miles between Adjacent Towns					
Town	**A**	**B**	**C**	**D**	**E**	**Destination**
Origin	40	60	50	—	—	—
A		10	—	70	—	—
B			20	55	40	—
C				—	50	—
D					10	60
E						80

Homework **3)** Find shortest path from O to T, first visually then using then using the table method and backward recursion studied in Lesson 4 (Hillier 10.3-4); first row of the table below.



n	Solved Nodes Directly Connected to Unsolved Nodes	Closest Connected Unsolved Node	Total Distance Involved	nth Nearest Node	Minimum Distance	Last Connection
1	*O*	*A*	4	*A*	4	*OA*

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Homework

4) Go back to eCampus Lesson three slides 55 and 56 about type one and type two error – or read about them online. Make an example of a test setting and describe for that test what would be type 1 and type two errors and the respective implications.

Thank you

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https://www.youtube.com/channel/UCz26ZK04xchekUy4GevA3DA

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