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COIN

Non compensatory Multicriteria Methods

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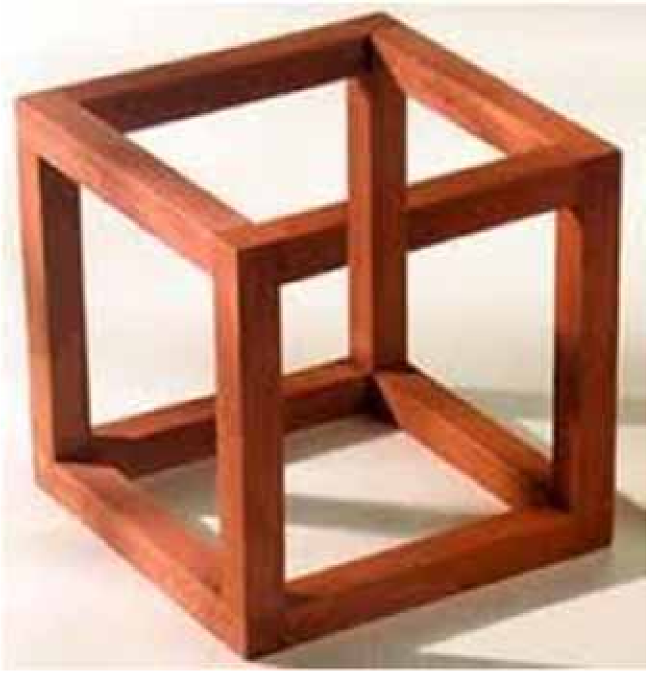
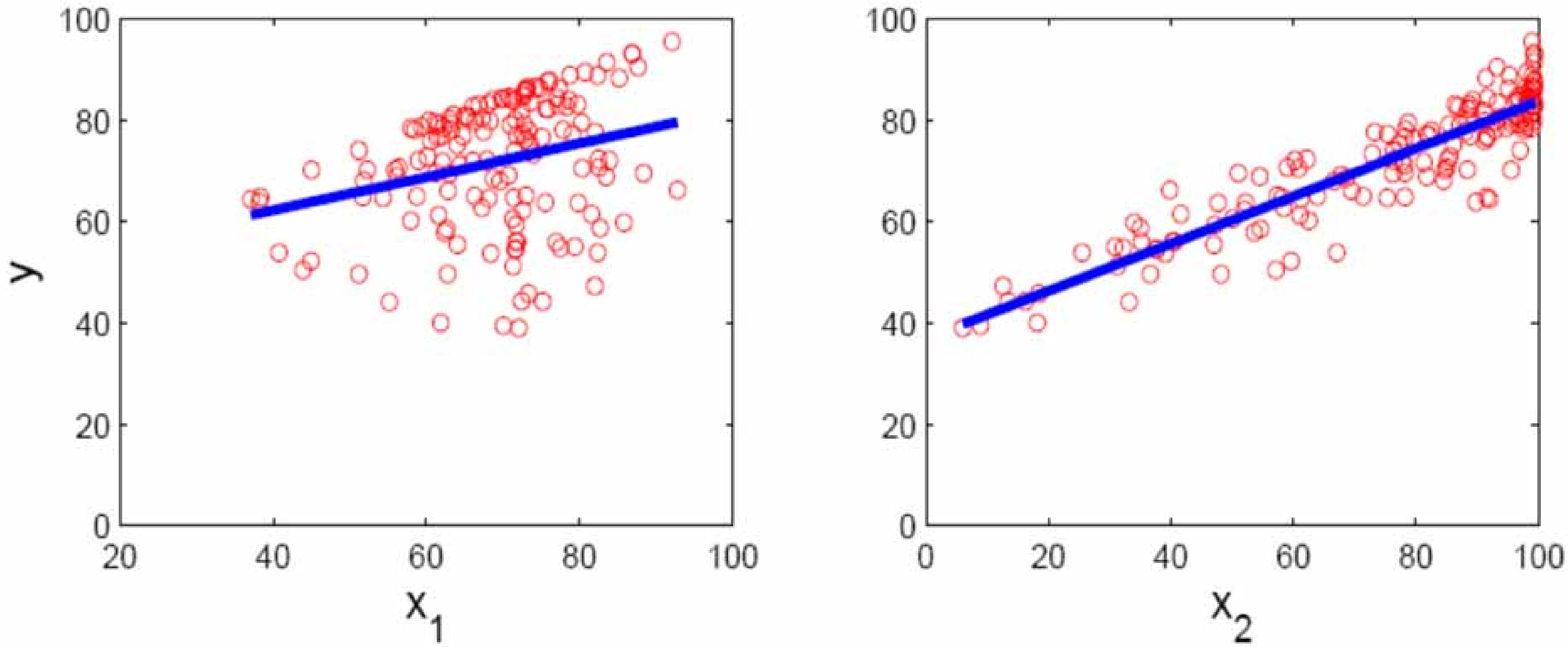
There is a theoretical inconsistency in the way weights are used in practice (e.g. in linear aggregation) and their real theoretical meaning.

The ‘Dean Example’.

The linear aggregation paradox: weights are used as if they were importance coefficients while they are trade off coefficients



An example. A dean wants to rank teachers based on 'hours of teaching' and 'number of publications'. Unfortunately when adding these two variables up she sees that <u>teachers are practically ranked by publications.</u>



Dean's example: $y=x_1+x_2$.

Estimated $R_1^2 = 0.0759$, $R_2^2 = 0.826$,

$corr(x_1, x_2) = -0.151$, $V(x_1) = 116$, $V(x_2) = 614$, $V(y) = 162$.

X_1: hours of teaching X_2: number of publications

To obviate this the dean substitutes the model

$y=1/2(x_1+x_2)$

with

$y=0.7x_1+0.3x_2$

X_1: hours of teaching
X_2: number of publications

A professor comes by, looks at the last formula, and complains that publishing is disregarded in the department …


We offer three kinds of service:
GOOD - CHEAP - FAST
You can pick any two
GOOD service CHEAP won't be FAST
GOOD service FAST won't be CHEAP
FAST service CHEAP won't be GOOD

- Linear aggregation only possible under special circumstances (eg standardized variables, uniform covariance matrix…)

There is a theoretical inconsistency in the way weights are used in practice (e.g. in linear aggregation) and their real theoretical meaning.

For the weights to be interpreted as "*importance coefficients* " (the greatest weight is placed on the most important "dimension") non-compensatory aggregation procedures must be used to construct composite indicators.

MCA problem

In a multi-criteria problem, there is no solution optimising all the criteria at the same time (the so-called ideal or utopia solution) and therefore compromise solutions have to be found.

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The aggregation of several criteria implies taking a position on the fundamental issue of compensability.

Compensability refers to the existence of trade-offs, i.e. the possibility of offsetting a disadvantage on some criteria by an advantage on another criterion.

Compensability

E.g. in the construction of a composite indicator of human development a compensatory logic (using equal weighting) would imply that one is willing to accept 10% less enrolling in secondary education in exchange of a 10% increase in GDP.

Social choice theory methods

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Social choice theory methods would be ideally suited for building meaningful aggregated indicators … and were already available between the end of the XIII and the XV century, …

… but almost everybody uses linear aggregation

Notes to the previous page

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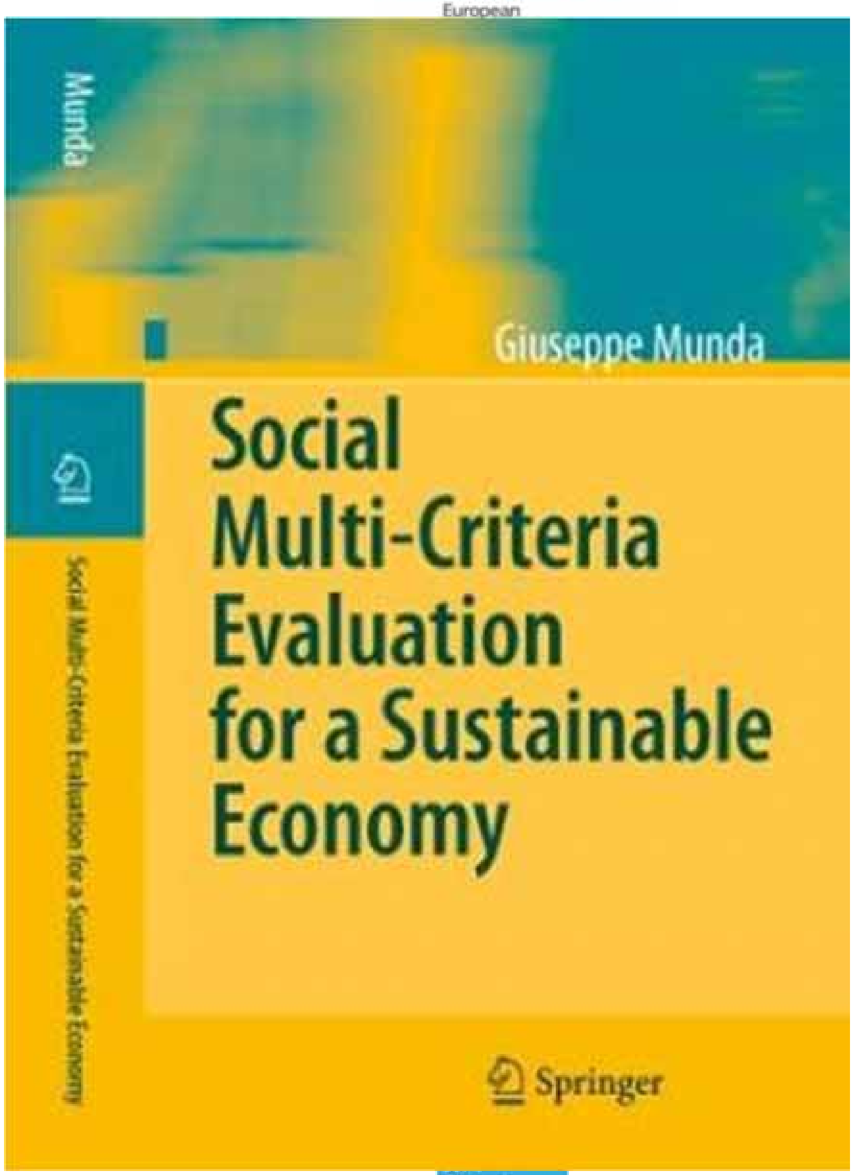
Ramon Llull (ca. 1232 – ca. 1315) proposed first what would then become known as the method of Condorcet.

Nicholas of Kues (1401 – August 11, 1464), also referred to as Nicolaus Cusanus and Nicholas of Cusa developed what would later be known as the method of Borda.

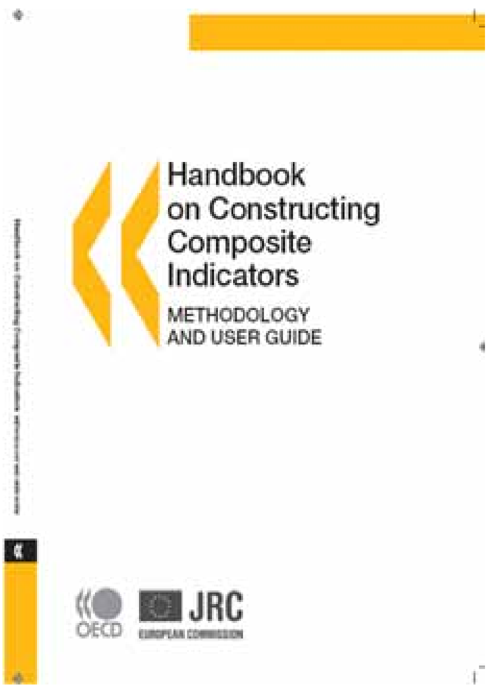
Nicolas de Condorcet, (17 September 1743 – 28 March 1794) developed the method eponimous. His 'Sketch for a Historical Picture of the Progress of the Human Spirit (1795)' can be considered as an ideological foundation for evidence based policy (modernity at its best!).

Jean-Charles, chevalier de Borda (May 4, 1733 – February 19, 1799) developed the Borda count.

All examples and discussion based on:



See also:



Step 1. Developing a theoretical framework

Step 2. Selecting indicators

Step 3. Multivariate analysis

Step 4. Imputation of missing data

Step 5. Normalisation of data

Step 6. Weighting and aggregation

Step 7. Robustness and sensitivity

Step 8. Association with other variables

Step 9. Back to the details (indicators)

Step 10. Presentation and dissemination

Social choice theory = Multi Criteria Analysis

(see Social Choice and Multi Criteria Decision Making by Kenneth Arrow and Herve' Raynaud, 1986).

But the same theory also works for building composite indicators.

Voters voting for candidates are equivalent to …

… Criteria 'voting' for options which is equivalent to …

….Indicators 'voting' for countries

An impact matrix

	Indic.	GDP ↑	Unemp. Rate ↓	Solid wastes ↓	Income dispar. ↓	Crime rate ↓
Country						
A		**25,000**	**0.15**	**0.4**	**9.2**	**40**
B		**45,000**	**0.10**	**0.7**	**13.2**	**52**
C		**20,000**	**0.08**	**0.35**	**5.3**	**80**
weights		**.166**	**.166**	**0.333**	**.166**	**.166**

We can say that indicator GDP 'votes' for B>A>C (countries / options)

What information can be derived from the impact matrix?

- Intensity of preference (the value of the indicator/criterion)
- Number of indicators/criterions in favour of a given country/option

What information can be derived from the impact matrix?

- Weight attached to each indicator/criterion
- Relative Performance of each country/option with respect to each of the other countries/option

Combinations of this ingredients generate different aggregation conventions

	Indic.	GDP	Unemp. Rate	Solid wastes	Income dispar.	Crime rate
Country						
A		**25,000**	**0.15**	**0.4**	**9.2**	**40**
B		**45,000**	**0.10**	**0.7**	**13.2**	**52**
C		**20,000**	**0.08**	**0.35**	**5.3**	**80**
weights		**.166**	**.166**	**0.333**	**.166**	**.166**

# of indicators	2	1	1	1
1st position	*c*	*b*	*c*	*a*
2nd position	*a*	*a*	*b*	*b*
3rd position	*b*	*c*	*a*	*c*

GDP:	B>A>C
UR:	C>B>A
SW:	C>A>B
ID:	C>A>B
CR:	A>B>C

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Different ways to organize the same information:

# of indicators	2	1	1	1
1st position	*c*	*b*	*c*	*a*
2nd position	*a*	*a*	*b*	*b*
3rd position	*b*	*c*	*a*	*c*

→

Rank	*a*	*b*	*c*		
1st	1	1	3		
2nd	3	2	0		
3rd	1	2	2		

Three countries [options/candidates] and five indicators [criteria/voters]

Objective: find best country

A first possibility:
apply the *plurality rule* ➔ the country which is more often ranked in the first position is the winning one.
➔ Country *a* is the best (8/21).

# of indicators	3	5	7	6
1st position	*a*	*a*	*b*	*c*
2nd position	*b*	*c*	*d*	*b*
3rd position	*c*	*b*	*c*	*d*
4th position	*d*	*d*	*a*	*a*

BUT Country *a* is also the one with the strongest opposition since 13/21 indicators put it into the last position!

This paradox was the starting step of Borda's and Condorcet's research at the end of the 18th century, but the plurality rule corresponds to the most common electoral system in the 21st century!

Two main lessons can be learned from the plurality rule paradox:

- ✓ Good ranking procedures should consider the whole ranking of countries and not the first position only.
- ✓ It is important to consider not only what a majority of indicators prefers but also what they do not prefer at all.

Borda & Condorcet solution to the plurality rule paradox next…

Frequency matrix (Moulin, 21 criteria 4 options)

Jean-Charles,
chevalier de Borda

Rank	*a*	*b*	*c*	*d*	Points
1st	**8**	**7**	**6**	**0**	**3**
2nd	**0**	**9**	**5**	**7**	**2**
3rd	**0**	**5**	**10**	**6**	**1**
4th	**13**	**0**	**0**	**8**	**0**

Columns add up to the number of criteria/voters=21

How was the frequency matrix generated?

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# of indicators	3	5	7	6
1st position	*a*	*a*	*b*	*c*
2nd position	*b*	*c*	*d*	*b*
3rd position	*c*	*b*	*c*	*d*
4th position	*d*	*d*	*a*	*a*

Rank	*a*	*b*	*c*	*d*	Points
1st	8	7	6	0	3
2nd	0	9	5	7	2
3rd	0	5	10	6	1
4th	13	0	0	8	0

Frequency matrix
(21 criteria 4 alternatives)

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Borda score:

$$a = 8 \times 3 = 24$$
$$b = 5 + 9 \times 2 + 7 \times 3 = 44$$
$$c = 10 + 5 \times 2 + 6 \times 3 = 38$$
$$d = 6 + 7 \times 2 = 20$$

Borda solution:
b→c→a→d

Now: country *b* is the best, no longer *a*, which was the winner when applying the plurality rule; The plurality rule paradox has been solved.

Rank	*a*	*b*	*c*	*d*	Points
1st	**8**	**7**	**6**	**0**	**3**
2nd	**0**	**9**	**5**	**7**	**2**
3rd	**0**	**5**	**10**	**6**	**1**
4th	**13**	**0**	**0**	**8**	**0**

Frequency matrix
(21 criteria 4 alternatives)

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The Borda count was developed independently several times, (e.g. by Nicolaus Cusanus beginning XV century) but is named for Jean-Charles de Borda, who devised the system in 1770.

It is currently used for the election of two ethnic minority members of the National Assembly of Slovenia (source: Wikipedia En). It is used throughout the world by various organisations and competitions [e.g. in academia].

Borda was a mariner and a scientist. Worked on chronometers. Between 1777 and 1778, he participated in the American Revolutionary War.

The French Academy of Sciences used Borda's method to elect its members for about two decades [till Napoleon Bonaparte became president…].

(17 September 1743 – 28 March 1794), known as Nicolas de Condorcet, was a philosopher, mathematician, and early political scientist […] advocated a liberal economy, free and equal public education, […] equal rights for women […] He died a mysterious death during the French revolution.

# of indicators	3	5	7	6
1st position	*a*	*a*	***b***	***c***
2nd position	***b***	***c***	***d***	***b***
3rd position	***c***	***b***	***c***	***d***
4th position	***d***	***d***	*a*	*a*

Frequency matrix

$$\begin{bmatrix} & a & b & c & d \\ a & 0 & 8 & 8 & 8 \\ b & 13 & 0 & 10 & 21 \\ c & 13 & 11 & 0 & 14 \\ d & 13 & 0 & 7 & 0 \end{bmatrix}$$

Outscoring matrix

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# of indicators	3	5	7	6
1st position	*a*	*a*	*b*	*c*
2nd position	*b*	*c*	*d*	*b*
3rd position	*c*	*b*	*c*	*d*
4th position	*d*	*d*	*a*	*a*

Frequency matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\left[\begin{array}{cccc} a & b & c & d \\ 0 & 8 & 8 & 8 \\ 13 & 0 & 10 & 21 \\ 13 & 11 & 0 & 14 \\ 13 & 0 & 7 & 0 \end{array}\right]$$

Outscoring matrix

For each pair of countries a concordance index is computed by counting how many indicators/voters are in favour of each country (e.g. 13 voters prefer b to a).

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 8 & 8 & 8 \\ b & 13 & 0 & 10 & 21 \\ c & 13 & 11 & 0 & 14 \\ d & 13 & 0 & 7 & 0 \end{array}$$

Outranking matrix

Note the "constant sum property" in the outranking matrix (13+8=21 number of indicators/voters)

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 8 & 8 & 8 \\ b & 13 & 0 & 10 & 21 \\ c & 13 & 11 & 0 & 14 \\ d & 13 & 0 & 7 & 0 \end{array}$$

Pairs with concordance index > 50% of the indicators/voters are considered: majority threshold = 11 (i.e. a number of voters > 50% of voters=21)

Thus *bPa= 13, bPd=21(=always), cPa=13, cPb=11, cPd=14, dPa=13.*

Count row-wise discarding entries < 11 as there are 21voters/criteria

c is better than a,b,d so it is the winner

b is better than the remaining a,d, it is the second best

d is better than a.

➔Condorcet solution: c → b→ d→ a

Both Borda and Condorcet approaches solve the plurality rule paradox. However, the solutions offered are different.

Borda solution: b → c → a → d
Condorcet solution: c → b → d → a

In the framework of composite indicators, can we choose between Borda and Condorcet on some theoretical and/or practical grounds?

Some difficulties …

Example with 3 options, 60 indicators/voters [Condorcet's own example, 1785]

Number of indicators	23	17	2	10	8
1st	*a*	*b*	*b*	*c*	*c*
2nd	*b*	*c*	*a*	*a*	*b*
3rd	*c*	*a*	*c*	*b*	*a*

Borda approach

Frequency matrix

Rank	*a*	*b*	*c*	Points
1st	**23**	**19**	**18**	**2**
2nd	**12**	**31**	**17**	**1**
3rd	**25**	**10**	**25**	**0**

Borda count:

$$a = 58,\ b = 69,\ c = 53$$

Condorcet approach

Outranking matrix

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 33 & 25 \\ b & 27 & 0 & 42 \\ c & 35 & 18 & 0 \end{array}$$

Concordance threshold =31

aPb, bPc and cPa (cycle)???

From this example we might conclude that the Borda rule is more effective since a country is always selected while the Condorcet one sometimes leads to an irreducible indecisiveness.

However Borda rules have other drawbacks, too

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Borda score:

$$a = 13,\ b = 12,\ c = 11,\ d = 6$$

a → b → c → d

So far, so good …

Rank	*a*	*b*	*c*	*d*	Points
1st	**2**	**2**	**3**	**0**	**3**
2nd	**2**	**3**	**0**	**2**	**2**
3rd	**3**	**0**	**2**	**2**	**1**
4th	**0**	**2**	**2**	**3**	**0**

Taking away an irrelevant country/option (d) a new frequency matrix is generated

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Rank	*a*	*b*	*c*	*d*	Points
1st	2	2	3	0	3
2nd	2	3	0	2	2
3rd	3	0	2	2	1
4th	0	2	2	3	0

a → b→ c → d

Rank	*a*	*b*	*c*	Points
1st	2	2	3	2
2nd	2	3	0	1
3rd	3	0	2	0

a=6,b=7,c=6

Now: b→ (c,a) … i.e. a reversal !

Unfortunately, Borda rule is fully dependent on irrelevant options and preference reversals can happen with an extreme high frequency

Implications for football

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Disqualifying a ‘poor’ club may result in a different team winning the championship!

Given that there is a consensus in the literature that the Condorcet' theory of voting is non-compensatory while Borda's one is compensatory, when one wishes to have weights as importance coefficients then Condorcet should be used.

Basic problem: presence of cycles, i.e. *aPb*, *bPc* and *cPa*

The probability of obtaining a cycle increases with both N. of indicators) and N. of countries

Condorcet himself was aware of this problem (he built examples to explain it) and he was even close to find a consistent rule able to rank any number of alternatives when cycles are present…

… but

Basic problem: presence of cycles, i.e. aPb, bPc and cPa

Furter attempts made by Kemeny (1959) and by Young and Levenglick (1978) … led to:

Condorcet-Kemeny-Young-Levenglick (C-K-Y-L) ranking procedure

Main methodological foundation: maximum likelihood concept.

The maximum likelihood principle selects as a final ranking the one with the maximum pair-wise support.

What does this mean and how does it work?

C-K-Y-L ranking procedure

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We go back to Condorcet's own example, 60 criteria 3 alternatives ➔ concordance threshold =31

Based on the Cordorcet:
aPb 33, bPc 42 and cPa 35 (cycle) .

Outranking matrix

$$
\begin{array}{c|ccc}
 & a & b & c \\
\hline
a & 0 & 33 & 25 \\
b & 27 & 0 & 42 \\
c & 35 & 18 & 0
\end{array}
$$

Outranking matrix

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{bmatrix} a & b & c \\ 0 & 33 & 25 \\ 27 & 0 & 42 \\ 35 & 18 & 0 \end{bmatrix}$$

But *bPa=27, cPb=18 and aPc=25* were not used as below the concordance threshold. Let us use them now:

Support of permutation abc =
=aPb+aPc+bPc =33+25+42=100

Support of permutation bca=
=bPc+bPa+cPa = 42+27+35=104

and so on for all 3! permutations of three elements a,b,c

Computing the support for all permutations

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$$\begin{bmatrix} & a & b & c \\ a & 0 & 33 & 25 \\ b & 27 & 0 & 42 \\ c & 35 & 18 & 0 \end{bmatrix}$$

Outranking matrix

a	***b***	***c***	**100**
b	*c*	*a*	104
c	***a***	***b***	**86**
b	***a***	***c***	**94**
c	***b***	***a***	**80**
a	***c***	***b***	**76**

Support

In conclusion:

$$b \rightarrow c \rightarrow a$$

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{bmatrix} a & b & c \\ 0 & 33 & 25 \\ 27 & 0 & 42 \\ 35 & 18 & 0 \end{bmatrix}$$

a	***b***	***c***	**100**
b	*c*	*a*	104
c	***a***	***b***	**86**
b	***a***	***c***	**94**
c	***b***	***a***	**80**
a	***c***	***b***	**76**

$$b \rightarrow c \rightarrow a$$

Is the solution with the highest support – likelihood

The original Condorcet problem has been solved in a satisfactory way by the C-K-Y-L.

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	Indic.	GDP	Unemp. Rate	Solid wastes	Income dispar.	Crime rate
Country		↑	↓	↓	↓	↓
A		**25,000**	**0.15**	**0.4**	**9.2**	**40**
B		**45,000**	**0.10**	**0.7**	**13.2**	**52**
C		**20,000**	**0.08**	**0.35**	**5.3**	**80**
weights		**.166**	**.166**	**0.333**	**.166**	**.166**

When is A>B?

On GDP? No
On unempl.? No
On waste? Yes
On disparity? Yes
On crime? Yes

AB = 0.333+0.166+0.166=0.666

BA = 0.166+0.166=0.333

AC = 0.166+0.166=0.333

CA = 0.166+0.333+0.166=0.666

BC = 0.166+0.166=0.333

CB = 0.166+0.333+0.166=0.666

C-K-Y-L ranking procedure

	Indic.	GDP	Unemp. Rate ↑	Solid wastes ↓	Income dispar. ↓	Crime rate ↓
Country						
A		25,000	0.15	0.4	9.2	40
B		45,000	0.10	0.7	13.2	52
C		20,000	0.08	0.35	5.3	80
weights		.166	.166	0.333	.166	.166

	A	B	C
A	0	0.666	0.333
B	0.333	0	0.333
C	0.666	0.666	0

AB = 0.333+0.166+0.166=0.666

BA = 0.166+0.166=0.333

AC = 0.166+0.166=0.333

CA = 0.166+0.333+0.166=0.666

BC = 0.166+0.166=0.333

CB = 0.166+0.333+0.166=0.666

C-K-Y-L ranking procedure

	Indic.	GDP	Unemp. Rate	Solid wastes	Income dispar.	Crime rate
Country		↑	↓	↓	↓	↓
A		25,000	0.15	0.4	9.2	40
B		45,000	0.10	0.7	13.2	52
C		20,000	0.08	0.35	5.3	80
weights		.166	.166	0.333	.166	.166

	A	B	C
A	0	0.666	0.333
B	0.333	0	0.333
C	0.666	0.666	0

BCA = 0.333 + 0.333 + 0.666 = 1.333

ABC = 0.666 + 0.333 + 0.333 = 1.333

CAB = 0.666 + 0.666 + 0.666 = 2

ACB = 0.333 + 0.666 + 0.666 = 1.666

CBA = 0.666 + 0.333 + 0.666 = 1.666

BAC = 0.333 + 0.333 + 0.333 = 1

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The Computational problem

The only drawback of this aggregation method is the difficulty in computing it when the number of candidates grows.

With only 10 countries → 10! = 3,628,800 permutations (instead of 3!=6 of the example)

To solve this problem one needs to use numerical algorithms (see references)

Related Reading

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